
Preliminaries

```
In[1]:= h[x_] := 1 / Pi (Sqrt[1 - x^2] - x ArcCos[x]);
G[λ_, z_] := λ h[z / λ];
Wdisk[d_, λ_] := (λ / 2)^d Gamma[1 + d / 2]^(-2);
κ[d_, m_] := If[m == 0, 1, Binomial[m + d - 1, d - 1] - Binomial[m + d - 3, d - 1]];
```

Verified approximations

```
In[5]:= CosUp[x_] := 1 -  $\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + \frac{x^{12}}{479001600}$ ;
CosDown[x_] := 1 -  $\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + \frac{x^{12}}{479001600} - \frac{x^{14}}{87178291200}$ ;
VerifiedQUpArcCos[x_, ε_] := Module[{ac0, ac},
  ac0 = ArcCos[x];
  If[MatchQ[ac0, _Rational] || IntegerQ[ac0], ac = ac0,
    ac = Rationalize[ac0 + 2 ε, ε];
    If[CosUp[ac] ≥ x, Message[VerifiedQUpArcCos::Error, x], ac]];
  (* ac=Rationalize[ArcCos[x]+2ε,ε]; *)
  ac
];
VerifiedQUpArcCos::Error = "ArcCosUp of argument `1` is wrong!";
VerifiedQDownArcCos[x_, ε_] := Module[{ac0, ac},
  ac0 = ArcCos[x];
  If[MatchQ[ac0, _Rational] || IntegerQ[ac0], ac = ac0,
    ac = Rationalize[ac0 - 2 ε, ε];
    If[CosDown[ac] ≤ x, Message[VerifiedQDownArcCos::Error, x], ac]];
  ac
];
VerifiedQDownArcCos::Error = "ArcCosDown of argument `1` is wrong!";
VerifiedQUpSqrt[x_, ε_] := Module[{ac0, ac},
  ac0 = Sqrt[x];
  If[MatchQ[ac0, _Rational] || IntegerQ[ac0], ac = ac0,
    ac = Rationalize[ac0 + 2 ε, ε];
    If[ac^2 < x, Message[VerifiedQUpSqrt::Error, x], ac]];
  ac
];
VerifiedQUpSqrt::Error = "SqrtUp of argument `1` is wrong!";
VerifiedQDownSqrt[x_, ε_] := Module[{ac0, ac},
  ac0 = Sqrt[x];
  If[MatchQ[ac0, _Rational] || IntegerQ[ac0], ac = ac0,
    ac = Rationalize[ac0 - 2 ε, ε];
    If[ac^2 > x, Message[VerifiedQDownSqrt::Error, x], ac]];
  ac
];
```

```

];
VerifiedQDownSqrt::Error = "SqrtDown of argument `1` is wrong!";
VerifiedQUpRoot[x_, d_, ε_] := Module[{ac0, ac},
  ac0 = x^(1/d);
  If[MatchQ[ac0, _Rational] || IntegerQ[ac0], ac = ac0,
    ac = Rationalize[ac0 + 2 ε, ε];
    If[ac^d < x, Message[VerifiedQUpRoot::Error, x], ac]];
  ac
];
VerifiedQUpRoot::Error = "RootUp `2` of argument `1` is wrong!";
VerifiedQDownRoot[x_, d_, ε_] := Module[{ac0, ac},
  ac0 = x^(1/d);
  If[MatchQ[ac0, _Rational] || IntegerQ[ac0], ac = ac0,
    ac = Rationalize[ac0 - 2 ε, ε];
    If[ac^d > x, Message[VerifiedQDownRoot::Error, x], ac]];
  ac
];
VerifiedQDownSqrt::Error = "RootDown `2` of argument `1` is wrong!";

```

Algorithm

```

In[19]:= PiUp[ε_] := 3 VerifiedQUpArcCos[1/2, ε];
PiDown[ε_] := 3 VerifiedQDownArcCos[1/2, ε];
GQDown[λ_, z_, ε_] :=
  (VerifiedQDownSqrt[λ^2 - z^2, ε] - z VerifiedQUpArcCos[z/λ, ε]) / PiUp[ε];
WdiskQUp[d_, λ_, ε_] := Wdisk[d, λ] /. Pi → PiDown[ε];
adisk[d_, ε_] := WdiskQUp[d, λ, ε] / λ^d;
PNeuQDown[d_, λ_, ε_] := Total[Table[
  κ[d, m] Floor[GQDown[λ, m + d/2 - 1, ε] + 3/4], {m, 0, Floor[λ - d/2 + 1]}]];

```

```

In[25]:= StepCheck[d_, λ_, ε_] := Module[{e, λnew},
  e = Total[Table[κ[d, m] Floor[GQDown[λ, m + d / 2 - 1, ε] + 3 / 4],
    {m, 0, Floor[λ - d / 2 + 1]}]] - WdiskQUp[d, λ, ε];
  If[e ≤ 0, Print["Process stopped with e=", e]; Return[]];
  λnew = VerifiedQDownRoot[λ^d + e / adisk[d, ε], d, ε];
  {λ, λ // N, e, e // N, λnew, λnew // N}
];

RunChecks[d_, Δstart_, Δend_, ε_, printstep_] := Module[{Δ, nI, data, step},
  data = {};
  nI = 1;
  Δ = Δstart;
  Print["d=" <> ToString[d]];
  Do[
    step = StepCheck[d, Δ, ε];
    If[printstep, Print[step]];
    AppendTo[data, step];
    If[step[[5]] ≤ Δ, Break[]];
    Δ = step[[5]];
    If[Δ ≥ Δend, Break[]];
  ,
    {nI, 1, 100 000}
  ];
  step = StepCheck[d, Δ, ε];
  If[printstep, Print[step]];
  AppendTo[data, step];
  data
];

```

Actual Calculation

```

In[31]:= Δ20short = 3;
Δ21short = 14;
data2dshort = RunChecks[2, 3, 14, 10^(-3), True];
tb2short = Table[{j, data2dshort[[j, 1]],
  data2dshort[[j, 3]], data2dshort[[j, 5]] - data2dshort[[j, 1]],
  {j, 1, Length[data2dshort]}} // TableForm

```

d=2

$$\left\{ 3, 3., \frac{3}{4}, 0.75, \frac{45}{13}, 3.46154 \right\}$$

$$\left\{ \frac{45}{13}, 3.46154, \frac{1355}{676}, 2.00444, \frac{76}{17}, 4.47059 \right\}$$

$$\left\{ \frac{76}{17}, 4.47059, \frac{868}{289}, 3.00346, \frac{164}{29}, 5.65517 \right\}$$

$$\left\{ \frac{164}{29}, 5.65517, \frac{3368}{841}, 4.00476, \frac{187}{27}, 6.92593 \right\}$$

$$\left\{ \frac{187}{27}, 6.92593, \frac{11687}{2916}, 4.00789, 8, 8. \right\}$$

$$\left\{ 8, 8., 3, 3., \frac{523}{60}, 8.71667 \right\}$$

$$\left\{ \frac{523}{60}, 8.71667, \frac{57671}{14400}, 4.00493, \frac{374}{39}, 9.58974 \right\}$$

$$\left\{ \frac{374}{39}, 9.58974, \frac{6098}{1521}, 4.0092, \frac{239}{23}, 10.3913 \right\}$$

$$\left\{ \frac{239}{23}, 10.3913, \frac{10591}{2116}, 5.0052, \frac{181}{16}, 11.3125 \right\}$$

$$\left\{ \frac{181}{16}, 11.3125, \frac{4103}{1024}, 4.00684, 12, 12. \right\}$$

$$\left\{ 12, 12., 6, 6., \frac{324}{25}, 12.96 \right\}$$

$$\left\{ \frac{324}{25}, 12.96, \frac{2506}{625}, 4.0096, \frac{217}{16}, 13.5625 \right\}$$

$$\left\{ \frac{217}{16}, 13.5625, \frac{7183}{1024}, 7.01465, \frac{495}{34}, 14.5588 \right\}$$

$$\left\{ \frac{495}{34}, 14.5588, \frac{27791}{4624}, 6.01016, \frac{384}{25}, 15.36 \right\}$$

Out[34]//TableForm=

1	3	$\frac{3}{4}$	$\frac{6}{13}$
2	$\frac{45}{13}$	$\frac{1355}{676}$	$\frac{223}{221}$
3	$\frac{76}{17}$	$\frac{868}{289}$	$\frac{584}{493}$
4	$\frac{164}{29}$	$\frac{3368}{841}$	$\frac{995}{783}$
5	$\frac{187}{27}$	$\frac{11687}{2916}$	$\frac{29}{27}$
6	8	3	$\frac{43}{60}$
7	$\frac{523}{60}$	$\frac{57671}{14400}$	$\frac{227}{260}$
8	$\frac{374}{39}$	$\frac{6098}{1521}$	$\frac{719}{897}$
9	$\frac{239}{23}$	$\frac{10591}{2116}$	$\frac{339}{368}$
10	$\frac{181}{16}$	$\frac{4103}{1024}$	$\frac{11}{16}$
11	12	6	$\frac{24}{25}$
12	$\frac{324}{25}$	$\frac{2506}{625}$	$\frac{241}{400}$
13	$\frac{217}{16}$	$\frac{7183}{1024}$	$\frac{271}{272}$
14	$\frac{495}{34}$	$\frac{27791}{4624}$	$\frac{681}{850}$