From asymptotics to bounds: recent progress on Pólya's eigenvalue conjecture

Michael Levitin

University of Reading
michaellevitin.net

ICBS, Beijing, 13th August 2026

based on joint works with

Nikolay Filonov	**Iosif Polterovich**	**David A. Sher**
		

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Spectral geometry

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain, $-\Delta := -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$, and consider the spectral problems for

Dirichlet Laplacian

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Neumann Laplacian ($\partial\Omega$ Lipschitz)

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$$(0 <)\lambda_1^{\mathrm{Dir}} \leq \lambda_2^{\mathrm{Dir}} \leq \cdots \leq \lambda_n^{\mathrm{Dir}} \leq \ldots, \qquad 0 = \lambda_1^{\mathrm{Neu}} \leq \lambda_2^{\mathrm{Neu}} \leq \cdots \leq \lambda_n^{\mathrm{Neu}} \leq \ldots$$

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Similar Dirichlet and Neumann spectral problems can be posed for Laplacian on a smooth compact Riemannian manifold with boundary.

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We always have $\lambda_n^{\mathrm{Neu}}(\Omega) < \lambda_n^{\mathrm{Dir}}(\Omega)$.

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Question/conjecture, early 20th century

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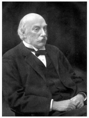

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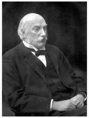

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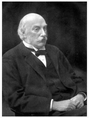

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One-term spectral asymptotics (1911)

$$\mathcal{N}_\Omega^{\aleph}(\lambda) = \underbrace{C_d |\Omega|_d \lambda^d}_{\text{Weyl's term}} + o\left(\lambda^d\right) \quad \text{as } \lambda \to \infty,$$

the Weyl constant $C_d := \dfrac{1}{(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}+1\right)}$, e.g. $C_2 = \dfrac{1}{4\pi}$

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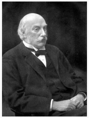

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Weyl's conjecture (1911)

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$$\mathcal{N}_{\Omega}^{\mathrm{Dir},\mathrm{Neu}}(\lambda) = C_d |\Omega|_d \lambda^d \mp C_{\mathrm{b},d} |\partial\Omega|_{d-1} \lambda^{d-1} + o\left(\lambda^{d-1}\right) \qquad \text{as } \lambda \to +\infty,$$

with $-$ for Dirichlet and $+$ for Neumann, and $C_{\mathrm{b},d} > 0$ another constant depending only on d.

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$$\mathcal{N}_\Omega^{\mathrm{Dir,Neu}}(\lambda) = \frac{\mathrm{Area}(\Omega)}{4\pi}\lambda^2 \mp \frac{\mathrm{Length}(\partial\Omega)}{4\pi}\lambda + o(\lambda) \qquad \text{as } \lambda \to +\infty.$$

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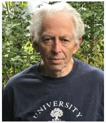
This turned out to be a difficult problem in analysis with connections to other fields.

Spectral asymptotics and billiard dynamics

Theorem [Ivrii'80]

The two-term Weyl asymptotics holds for any $\Omega \subset \mathbb{R}^d$ with piecewise-smooth boundary, satisfying the non-periodicity condition.



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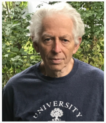

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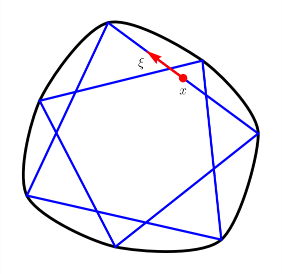

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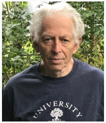

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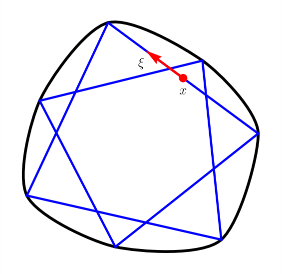

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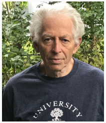

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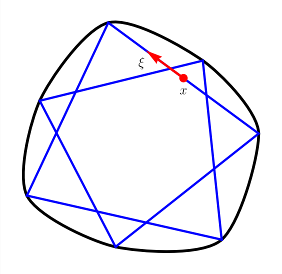

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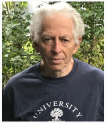

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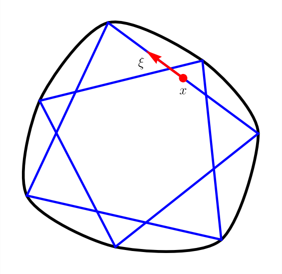

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Not true on some manifolds (in terms of geodesic flow), e.g. for a hemisphere.

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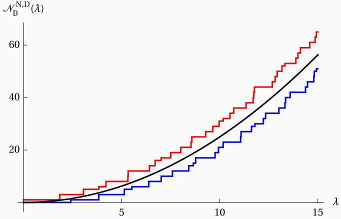

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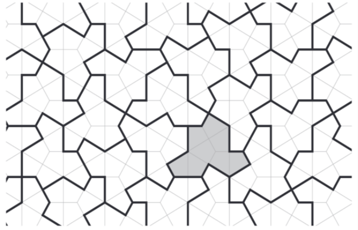

An einstein monotile, image from
D. Smith, J. S, Myers, C. S. Kaplan, and C. Goodman-Strauss,
An aperiodic monotile, Combinatorial Theory **4** (2024).

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- For all domains Ω [Berezin–Li–Yau, Kröger],

$$\left(\frac{d}{d+2}\right)^{d/2} \mathcal{N}_\Omega^{\mathrm{Dir}}(\lambda) \leq C_d|\Omega|_d\lambda^d \leq \frac{d+2}{2}\mathcal{N}_\Omega^{\mathrm{Neu}}(\lambda)$$

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- For low λ for any domain in any dimension:

$$\lambda_{n+1}^{\mathrm{Neu}}(\Omega) < \left(C_d |\Omega|_d\right)^{-2/d} n^{2/d} < \lambda_n^{\mathrm{Dir}}(\Omega)$$

with $n = 1$ or $n = 2$, by isoperimetric inequalities [Faber–Krahn, Krahn–Szegő, Girouard–Nadirashvili–Polterovich, Bucur–Henrot]

Gábor Szegő

FLPS 2023–26

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- an analogue holds for Aharonov–Bohm magnetic Dirichlet and Neumann Schrödinger operators in the disk

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FLPS 2023–26

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Why are disks, balls, etc. hard problems?

By separation of variables, eigenvalues can be found explicitly in terms positive zeros $j_{\nu,k}$ of Bessel functions J_ν or positive zeros $j'_{\nu,k}$ of their derivatives J'_ν, with $j'_{0,1} = 0$:

$m \in \{0, 1, \dots\}$	Dirichlet $\lambda^{\mathrm{Dir}}_{m,k}$	Neumann $\lambda^{\mathrm{Neu}}_{m,k}$	multiplicity
Disk $\mathbb{D} = \mathbb{B}^2$	$j^2_{m,k}$	${j'}^2_{m,k}$	$\begin{cases} 1, & m = 0, \\ 2, & m > 0 \end{cases}$
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$$\boxed{\mathcal{N}^{\mathrm{Dir}}_{\mathbb{D}}(\lambda) = \#\{k : j_{0,k} \leq \lambda\} + 2\sum_{m=1}^{\infty} \#\{k : j_{m,k} \leq \lambda\} < \frac{\lambda^2}{4}.}$$

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Neumann, unit disk: we need to prove that for all $\lambda > 0$,

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Phase functions for 2nd order ODEs

Let $\mathcal{A}_1(x)$ and $\mathcal{A}_2(x)$ be two real linearly independent solutions of a Schrödinger equation

$$\mathcal{A}''(x) + \mathcal{P}(x)\mathcal{A}(x) = 0 \tag{1}$$

on $(a, +\infty)$, with a real-valued potential $\mathcal{P} \in C(a, +\infty)$.

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We define the modulus function $\mathcal{M}(x) = \mathcal{M}_{\mathcal{A}_1,\mathcal{A}_2}(x)$ and the phase function $\Psi(x) = \Psi_{\mathcal{A}_1,\mathcal{A}_2}(x)$ for these solutions as

$$\mathcal{M}(x) := \sqrt{(\mathcal{A}_1(x))^2 + (\mathcal{A}_2(x))^2}, \qquad \mathcal{A}_1(x) + \mathrm{i}\mathcal{A}_2(x) = \mathcal{M}(x)\exp\left(\mathrm{i}\Psi(x)\right),$$

where we choose a continuous branch of $\Psi(x) = \operatorname{Arctan}\frac{\mathcal{A}_2(x)}{\mathcal{A}_1(x)}$ specified by the chosen value of $\lim_{x\to a^+}\Psi(x)$.

Example: Bessel phase functions

$$\theta_\nu(x) := \operatorname{Arctan}\frac{Y_\nu(x)}{J_\nu(x)}, \theta_\nu(0) := -\frac{\pi}{2}, \qquad \phi_\nu(x) := \operatorname{Arctan}\frac{Y'_\nu(x)}{J'_\nu(x)}, \phi_\nu(0) := \frac{\pi}{2}.$$

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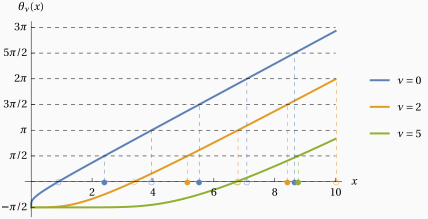

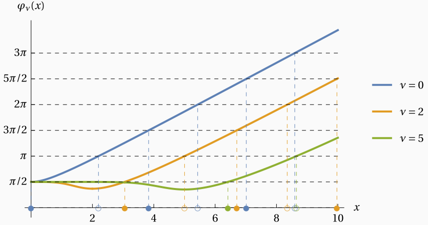

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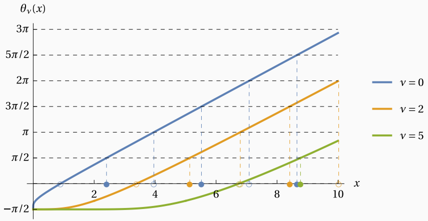

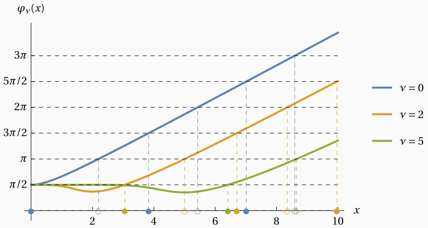

Both phase functions are non-oscillatory (monotone increasing) for $x \geq \nu$, hence

$$j_{\nu,k} = \theta_\nu^{-1}\left(\pi\left(k - \frac{1}{2}\right)\right) \qquad j'_{\nu,k} = \phi_\nu^{-1}\left(\pi\left(k - \frac{1}{2}\right)\right)$$

$$\#\{k : j_{\nu,k} \leq \lambda\} = \left\lfloor \frac{1}{\pi}\theta_\nu(\lambda) + \frac{1}{2} \right\rfloor \quad \#\{k : j'_{\nu,k} \leq \lambda\} = \left\lfloor \frac{1}{\pi}\phi_\nu(\lambda) + \frac{1}{2} \right\rfloor$$

From asymptotics to bounds: Bessel phase functions

A comparison theorem à la Sturm, an informal version

If one knows the asymptotics as $x \to \infty$ of a non-oscillating phase function $\Psi(x)$ of a Schrödinger equation, then, under some conditions and, possibly, with restrictions, one may be able to deduce uniform bounds for $\Psi(x)$.

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The asymptotics of Bessel phase function are well known, and we used them[2] to get

Uniform enclosures for Bessel phase functions

Define

$$G_\lambda(z) := \frac{1}{\pi}\left(\sqrt{\lambda^2 - z^2} - z \arccos\frac{z}{\lambda}\right), \qquad z \in [0, \lambda].$$

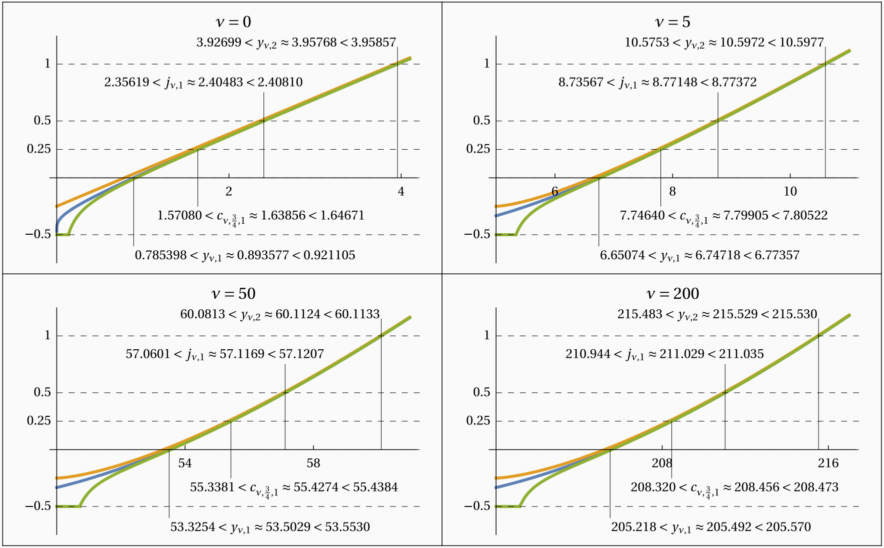
Then for all $x \geq \nu \geq 0$,

$$\cdots \leq \frac{1}{\pi}\theta_\nu(x) \leq G_x(\nu) - \frac{1}{4}, \qquad G_x(\nu) + \frac{1}{4} \leq \frac{1}{\pi}\phi_\nu(x) \leq \dots$$

(terms which are shown as $\cdots$ are explicit but not important for what follows).

[2] following D. A. Sher, Joint asymptotic expansions for Bessel functions. Pure Appl. Anal. **5** (2023)

Does it work? Yes, produces tight uniform enclosures!



Back to the disk

Combining

$$\mathcal{N}_{\mathbb{D}}^{\mathrm{Dir}}(\lambda) = \#\{k : j_{0,k} \leq \lambda\} + 2\sum_{m=1}^{\infty} \#\{k : j_{m,k} \leq \lambda\},$$

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Lattice point counts

The function $G_\lambda(z) := \frac{1}{\pi}\left(\sqrt{\lambda^2 - z^2} - z \arccos \frac{z}{\lambda}\right)$ is monotone decreasing, convex, and $\mathrm{Lip}_{\frac{1}{2}}$ on $[0, \lambda]$.

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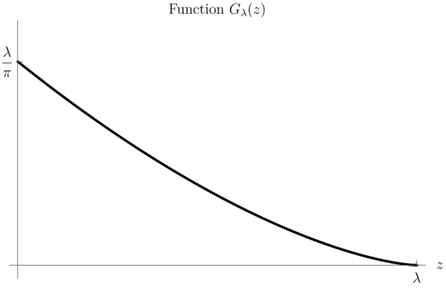

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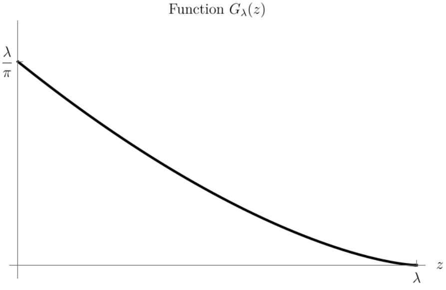
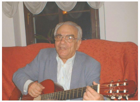


First introduced in [Kuznetsov–Fedosov'65], who used it to deduce improved $O\left(\lambda^{2/3}\right)$ remainder term estimate[a] in the two-term Weyl's Law for the disk.



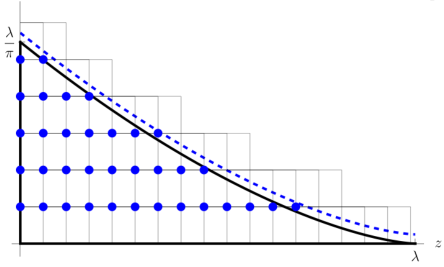
Boris Fedosov

[a] recently further improved in [Guo et al.]

Lattice point counts

We are counting (with weight, 1 for $z = 0$, 2 otherwise) lattice points under a shifted graph of $G_\lambda(z)$:

Dirichlet: weighted lattice point count under the graph of $G_\lambda(z) + \frac{1}{4}$



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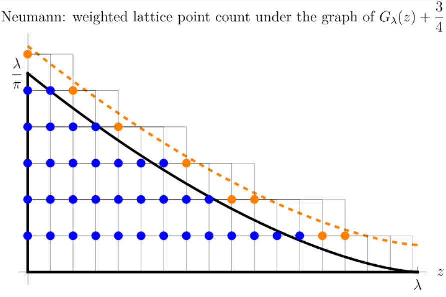

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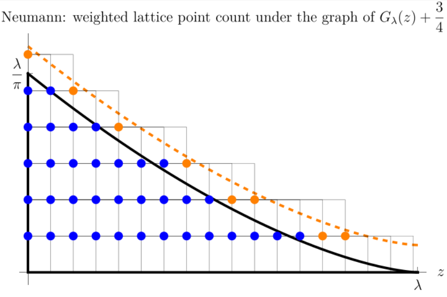

We want to compare the counts with

$$\frac{\lambda^2}{4} = 2\int_0^\lambda G_\lambda(z)\,\mathrm{d}z$$

Trapezoidal floor sums I

Definition

Let $a, b \in \mathbb{Z}$ with $a < b$, and let $g : [a, b] \to \mathbb{R}$. The trapezoidal floor sum of g on $[a, b]$ is defined as

$$\begin{aligned} \mathbf{T}(g, a, b) &= \frac{1}{2} \sum_{m=a}^{b-1} \left(\lfloor g(m) \rfloor + \lfloor g(m+1) \rfloor \right) \\ &= \frac{1}{2} \lfloor g(a) \rfloor + \sum_{m=a+1}^{b-1} \lfloor g(m) \rfloor + \frac{1}{2} \lfloor g(b) \rfloor . \end{aligned}$$

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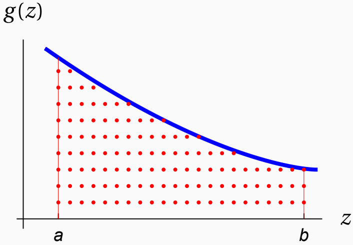
Obviously, $\mathbf{T}$ is additive:

$$\mathbf{T}(g, a, c) = \mathbf{T}(g, a, b) + \mathbf{T}(g, b, c), \qquad a < b < c.$$

Trapezoidal floor sums II



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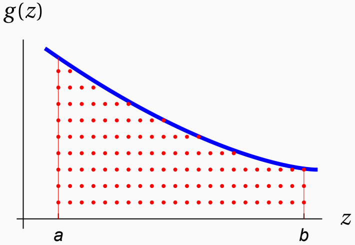
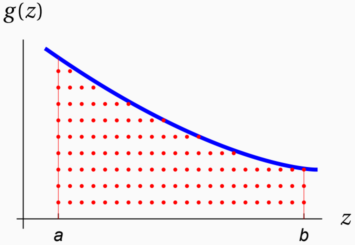


If g is non-negative, then $\mathbf{T}(g, a, b)$ counts the number of lattice points from $\mathbb{Z} \times \mathbb{N}$ lying on or under the graph of $g(z)$, but the lattice points with the edge abscissas a and b are counted with weight $\frac{1}{2}$.

Trapezoidal floor sums II



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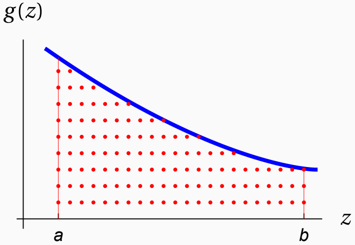

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Obtaining sharper asymptotics for this count as $\lambda \to \infty$ is a classical problem in number theory (van der Corput, etc.), but we need bounds.

From asymptotics to bounds: trapezoidal floor sums

We want to find conditions on g which guarantee an inequality between $\mathbf{T}(g, a, b)$ and $\int_a^b g(z)\,\mathrm{d}z$, maybe with correction terms. We always impose the following restrictions:

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$$\mathbf{T}\left(g + \frac{1}{4}, a, b\right) \leq \int_a^b g(z)\,\mathrm{d}z.$$

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Back to Pólya's conjecture for disks

Combining trapezoidal floor sums bounds with other estimates, we get:

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Extensions: higher-dimensional balls

- Dirichlet problem in $\mathbb{B}^d$, $d \geq 3$:
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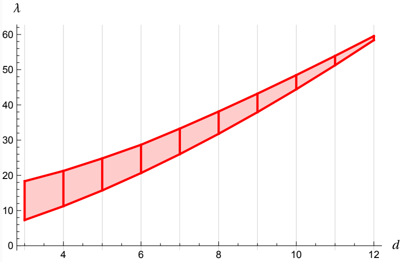
- Neumann problem in $\mathbb{B}^d$, $d \geq 3$:
  - eigenvalues relate to zeros of derivatives of ultraspherical Bessel functions
  - require extra combinatorial tricks, new variational bounds and harder calculus
  - fully analytic proof for all $\lambda > 0$ in dimensions $d \geq 12$; finite gaps in dimensions $3 \leq d \leq 12$ are filled by a computer-assisted algorithm



Extensions: Dirichlet problem in annuli I

For an annulus $A_r := \{x \in \mathbb{R}^2, r < |x| < 1\}$, Pólya's conjecture states that

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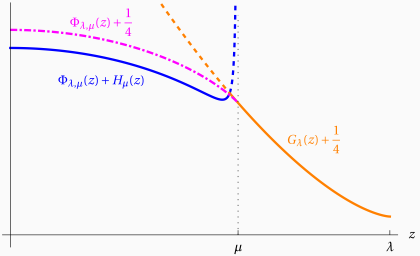
- Hence, sharper two-sided bounds for phase functions were needed.
- As a result, trapezoidal floor sums involve a mixture of concave and convex functions.



$$\mu := r\lambda$$

$$\Phi_{\lambda,\mu}(z) := G_\lambda(z) - G_\mu(z)$$

$$H_\mu(z) := \frac{3\mu^2 + 2z^2}{24\pi(\mu^2 - z^2)^{3/2}}, z \in [0,\mu)$$

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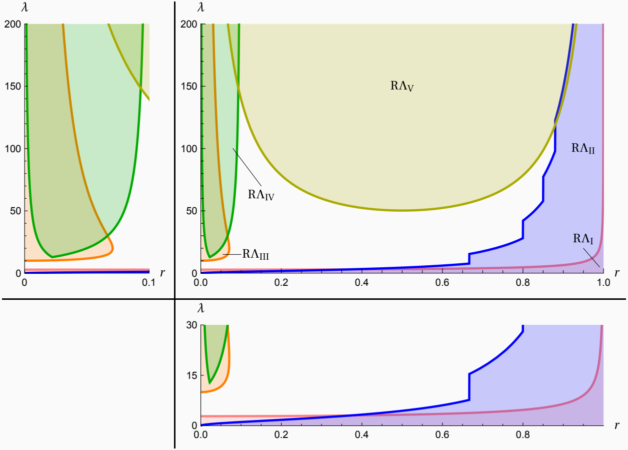
- Different analytic proofs in different regions, leaving a finite gap in $R\Lambda$, filled in using a rigorous computer-assisted algorithm



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- All scripts are available online

Other relevant recent papers I

- Improvements, extensions, different approaches, alternative proofs:
  - Y. Li, Pólya's conjecture for the Neumann eigenvalues on Euclidean balls. Preprint (2026). arXiv: 2607.25958.
  - Z. Gan, R. Jiang, and F.-H. Lin, The improved Berezin–Li–Yau inequality and Kröger inequality and consequences. Sci. China Math. (2026).
  - R. Jiang and F.-H. Lin, Pólya's conjecture up to ϵ-loss and quantitative estimates for the remainder of Weyl's law. Preprint (2025). arXiv: 2507.04307
  - N. Filonov, On the Pólya conjecture for the Neumann problem in planar convex domains. Comm. Pure Appl. Math. **78** (2025), no. 3, 537–544.
  - J. Guo, C. Miao, W. Wang, and G. Zhan, Improvement of Pólya's conjecture for balls and cylinders. Preprint (2025). arXiv: 2511.17050
- Partial/special cases, including problems on manifolds:
  - P. Freitas and I. Salavessa, Families of non-tiling domains satisfying Pólya's conjecture. J. Math. Phys. **64** (2023), no. 12, 121503.
  - X. He and Z. Wang, Pólya's conjecture for thin products, Preprint (2024). arXiv: 2402.12093
  - P. Freitas, J. Mao, and I. Salavessa, Pólya-type inequalities on spheres and hemispheres, Ann. Inst. Fourier **75** (2025).
  - D. Buoso, P. Luzzini, L. Provenzano, and J. Stubbe, Semiclassical estimates for eigenvalue means of Laplacians on spheres, J. Geom. Anal. **33** (2023).
  - P. Freitas, J. Lagacé, and J. Payette, Optimal unions of scaled copies of domains and Pólya's conjecture. Ark. Mat. **59** (2021), no. 1, 11–51.

Other relevant recent papers II

- Other related works:
  - R. L. Frank and S. Larson, Riesz means asymptotics for Dirichlet and Neumann Laplacians on Lipschitz domains. Invent. Math. **241** (2025), no. 3, 999–1079.
  - R. L. Frank and S. Larson, Semiclassical inequalities for Dirichlet and Neumann Laplacians on convex domains. Comm. Pure Appl. Math. **79** (2026), no. 3, 762–822.
  - J. Guo, T. Jiang, Z. Wang, and X. Yang, Bessel functions and Weyl's law for balls and spherical shells. Preprint (2024), to appear in Adv. Math. arXiv: 2412.14059

If you are interested in spectral geometry: adverts I

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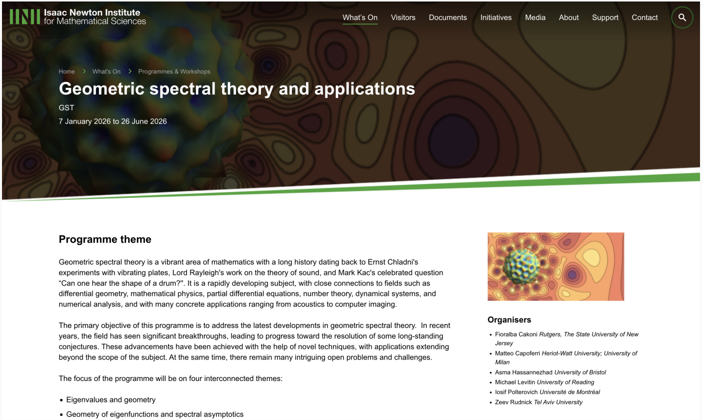
The webpage https://www.newton.ac.uk/event/gst/ of a recent INI programme has numerous videos of very interesting talks!



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Geometric spectral theory and applications

GST

7 January 2026 to 26 June 2026

Programme theme

Geometric spectral theory is a vibrant area of mathematics with a long history dating back to Ernst Chladni's experiments with vibrating plates, Lord Rayleigh's work on the theory of sound, and Mark Kac's celebrated question "Can one hear the shape of a drum?". It is a rapidly developing subject, with close connections to fields such as differential geometry, mathematical physics, partial differential equations, number theory, dynamical systems, and numerical analysis, and with many concrete applications ranging from acoustics to computer imaging.

The primary objective of this programme is to address the latest developments in geometric spectral theory. In recent years, the field has seen significant breakthroughs, leading to progress toward the resolution of some long-standing conjectures. These advancements have been achieved with the help of novel techniques, with applications extending beyond the scope of the subject. At the same time, there remain many intriguing open problems and challenges.

The focus of the programme will be on four interconnected themes:

- Eigenvalues and geometry
- Geometry of eigenfunctions and spectral asymptotics

Organisers

- Fioralba Cakoni *Rutgers, The State University of New Jersey*
- Matteo Capoferri *Heriot-Watt University; University of Milan*
- Asma Hassannezhad *University of Bristol*
- Michael Levitin *University of Reading*
- Iosif Polterovich *Université de Montréal*
- Zeev Rudnick *Tel Aviv University*

If you are interested in spectral geometry: adverts II

Our recent (AMS, 2023) book Topics in Spectral Geometry covers many topics related to Pólya's conjecture. A preliminary version is freely available at https://michaellevitin.net/Book/.

Michael Levitin
michaellevitin.net

Department of Mathematics and Statistics
University of Reading
M.Levitin@reading.ac.uk

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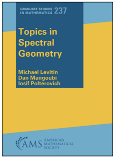
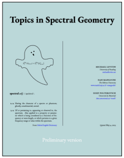
e: m.levitin@reading.ac.uk t: (+44)(0)(118) 378 8997 a: Pepper Lane, Whiteknights, Reading RG6 6AX, United Kingdom o: room 3.03

Topics in Spectral Geometry

Michael Levitin, Dan Mangoubi, and Iosif Polterovich

Published version	Preliminary version
AMS Graduate Studies in Mathematics series volume **237**, 2023.	May 29, 2023 large download; some differences with the final publisher's version
	
Erratum for published version (27 Jan 2025) Addendum for published version (27 Jan 2025)	Erratum for preliminary version (27 Jan 2025) Addendum for preliminary version (27 Jan 2025)

Download Mathematica and FreeFEM scripts for Appendix A

Thank you!