In[111]:=

```
SetDirectory[NotebookDirectory[]];
<< MaTeX`
texStyle = {};
SetOptions[MaTeX,
  "BasePreamble" → {"\\usepackage{amsmath}", "\\usepackage{xcolor}",
    "\\usepackage{fourier}", "\\usepackage{ebgaramond}"}, FontSize → 11];
```

In[115]:=

```
frame[legend_] :=
  Framed[legend, FrameStyle → Thin, RoundingRadius → 10, FrameMargins → 0];
```

In[116]:=

```
Clear[findAllRoots]
(* from https://
 mathematica.stackexchange.com/questions/16439/find-all-roots-of-an-
  interpolating-function-solution-to-a-differential-equation/16444#16444 *)
SyntaxInformation[findAllRoots] = {"LocalVariables" → {"Plot", {2, 2}},
   "ArgumentsPattern" → {_, _, OptionsPattern[]}};
SetAttributes[findAllRoots, HoldAll];

Options[findAllRoots] = Join[{"ShowPlot" → False, PlotRange → All},
   FilterRules[Options[Plot], Except[PlotRange]]];

findAllRoots[fn_, {l_, lmin_, lmax_}, opts : OptionsPattern[]] :=
 Module[{pl, p, x, localFunction, brackets},
  localFunction = ReleaseHold[Hold[fn] /. HoldPattern[l] :→ x];
  If[lmin ≠ lmax, pl = Plot[localFunction, {x, lmin, lmax}, Evaluate@
      FilterRules[Join[{opts}, Options[findAllRoots]], Options[Plot]]];
   p = Cases[pl, Line[{x__}] :→ x, Infinity];
   If[OptionValue["ShowPlot"],
    Print[Show[pl, PlotLabel → "Finding roots for this function",
      ImageSize → 200, BaseStyle → {FontSize → 8}]]], p = {}];
  brackets =
   Map[First, Select[(*This Split trick pretends that two points on the
       curve are "equal" if the function values have _opposite _ sign.Pairs
       of such sign-changes form the brackets for the subsequent FindRoot*)
     Split[p, Sign[Last[#2]] == -Sign[Last[#1]] &], Length[#1] == 2 &], {2}];
  x /. Apply[FindRoot[localFunction == 0, {x, ##1}] &, brackets, {1}] /. x → {}]
```

Coefficients of Weyl’s expansion in various settings

In[121]:=

```
αλ[λ_, μ_] := μ / (λ + 2 μ);
λα[α_, μ_] := μ (1 / α - 2);
CWeyl[d_, λ_, μ_] :=
  ((λ + 2 μ) ^ (-d / 2) + (d - 1) μ ^ (-d / 2)) / (4 Pi) ^ (d / 2) / Gamma[1 + d / 2];
BDirLiu[d_, λ_, μ_] :=
  - ((d - 1) / μ ^ ((d - 1) / 2) + 1 / (λ + 2 μ) ^ ((d - 1) / 2)) / 4 / (4 Pi) ^ ((d - 1) / 2) /
   Gamma[1 + (d - 1) / 2];
BDir[d_, α_, μ_] := -μ ^ ((1 - d) / 2) / (2 ^ (d + 1) Pi ^ ((d - 1) / 2) Gamma[(1 + d) / 2])
   (4 (d - 1) / Pi NIntegrate[τ ^ (d - 2) ArcTan[Sqrt[(1 - α τ ^ (-2)) (τ ^ (-2) - 1)]],
       {τ, Sqrt[α], 1}] + α ^ ((d - 1) / 2) + d - 1);
BDirSaVa2[α_, μ_] := 1 / (4 Pi Sqrt[μ]) (-1 - Sqrt[α] -
     4 / Pi NIntegrate[ArcTan[Sqrt[(1 - α / ξ ^ 2) (1 / ξ ^ 2 - 1)]], {ξ, Sqrt[α], 1}]);
BDirSaVa3[λ_, μ_] := -1 / (16 Pi) (3 λ ^ 2 + 13 λ μ + 16 μ ^ 2) / (λ ^ 2 μ + 5 λ μ ^ 2 + 6 μ ^ 3);
γR[α_] := Sqrt[Min[x /. Solve[x^3 - 8 x^2 + 16 (-1 + α) + 8 (3 - 2 α) x == 0, x, Reals]]];
Bfree[d_, α_, μ_] := μ ^ ((1 - d) / 2) / (2 ^ (d + 1) Pi ^ ((d - 1) / 2) Gamma[(1 + d) / 2])
   (4 (d - 1) / Pi NIntegrate[
       τ ^ (d - 2) ArcTan[(τ ^ (-2) - 2) ^ 2 / (4 Sqrt[(1 - α τ ^ (-2)) (τ ^ (-2) - 1)])],
       {τ, Sqrt[α], 1}] + α ^ ((d - 1) / 2) + d - 5 + 4 γR[α] ^ (1 - d));
BfreeSaVa2[α_, μ_] :=
  1 / (4 Pi Sqrt[μ]) (4 / γR[α] - 3 + Sqrt[α] + 4 / Pi NIntegrate[ArcTan[
        (2 - 1 / ξ ^ 2) ^ 2 / (4 Sqrt[(1 - α / ξ ^ 2) (1 / ξ ^ 2 - 1)])], {ξ, Sqrt[α], 1}]);
BfreeSaVa3[λ_, μ_] :=
  1 / (16 Pi) (3 (λ + 2 μ) ^ 2 - 3 (λ + 2 μ) μ + 2 μ ^ 2) / ((λ + 2 μ) μ (λ + μ));
```

Lame operator, L, is the operator of plane elasticity written in polar coordinates. For vectors, the fist component is along the radius.

In[132]:=

```
Lame =
  Function[u, (μ Curl[Curl[Flatten[{u, 0}], {r, ϕ, z}, "Cylindrical"], {r, ϕ, z},
        "Cylindrical"] - (λ + 2 μ) Grad[Div[Flatten[{u, 0}], {r, ϕ, z},
         "Cylindrical"], {r, ϕ, z}, "Cylindrical"])[[1 ;; 2]]];
Lame[{u1[r, ϕ], u2[r, ϕ]}]
```

Out[133]=

$$\left\{\frac{\mu\left(-\frac{-u_2^{(0,1)}[r,\phi]+u_1^{(0,2)}[r,\phi]}{r}+u_2^{(1,1)}[r,\phi]\right)}{r}-(\lambda+2\mu)\left(-\frac{u_1[r,\phi]+u_2^{(0,1)}[r,\phi]}{r^2}+\frac{u_1^{(1,0)}[r,\phi]+u_2^{(1,1)}[r,\phi]}{r}+u_1^{(2,0)}[r,\phi]\right),\right.$$
$$\left.-\frac{(\lambda+2\mu)\left(\frac{u_1^{(0,1)}[r,\phi]+u_2^{(0,2)}[r,\phi]}{r}+u_1^{(1,1)}[r,\phi]\right)}{r}+\mu\left(-\frac{-u_2[r,\phi]+u_1^{(0,1)}[r,\phi]}{r^2}+\frac{-u_2^{(1,0)}[r,\phi]+u_1^{(1,1)}[r,\phi]}{r}-u_2^{(2,0)}[r,\phi]\right)\right\}$$

The boundary conditions for the free boundary and for the Dirichlet problems are set up with

In[134]:=

```
T[u_] := ({λ Div[u, {r, ϕ}, "Polar"], 0} + 2 μ D[u, r] +
      μ ({1, 0, 0} × Curl[Flatten[{u, 0}], {r, ϕ, z}, "Cylindrical"])⟦1 ;; 2⟧ /.
     {r → 1}) // Simplify;
TDir[u_] := {u⟦1⟧, u⟦2⟧};
```

In[136]:=

```
T[{u1[r, ϕ], u2[r, ϕ]}]
TDir[{u1[r, ϕ], u2[r, ϕ]}]
```

Out[136]=

$$\left\{\lambda u_1[1,\phi]+\lambda u_2^{(0,1)}[1,\phi]+(\lambda+2\mu)\, u_1^{(1,0)}[1,\phi],\ \mu\left(-u_2[1,\phi]+u_1^{(0,1)}[1,\phi]+u_2^{(1,0)}[1,\phi]\right)\right\}$$

Out[137]=

$\{u_1[r,\phi], u_2[r,\phi]\}$

Solutions of Lame system for Λ=0: there are 4 solutions, {r,0}, {0,r}, {1, I} Exp[I ϕ], {1, -I} Exp[-I ϕ]
This is irrelevant for the discussion and can be skipped

In[138]:=

```
Collect[Lame[{a r, c r}], r, Simplify]
T[{a r, c r}]
```

Out[138]=

{0, 0}

Out[139]=

$\{2 a (\lambda+\mu), 0\}$

In[140]:=

```
Collect[Exp[-I ϕ] Lame[{b, I b} Exp[I ϕ]], r, Simplify]
T[{b, I b} Exp[I ϕ]]
```

Out[140]=

{0, 0}

Out[141]=

{0, 0}

In[142]:=

```
Collect[Exp[I ϕ] Lame[{b, -I b} Exp[-I ϕ]], r, Simplify]
T[{b, -I b} Exp[-I ϕ]]
```

Out[142]=

{0, 0}

Out[143]=

{0, 0}

In[144]:=

```
κ0 = (λ + 3 μ) / (λ + μ)
```

Out[144]=

$$\frac{\lambda+3\mu}{\lambda+\mu}$$

In[145]:=

```
ClearAll[soln0]; soln0[n_] := Which[
  n == 0, {{r, 0}, {0, r}},
  n == 1, {{1, I}, r^2 {(κ0 - 2), -I (κ0 + 2)}},
  n == -1, {{1, -I}, r^2 {(κ0 - 2), I (κ0 + 2)}},
  n ≥ 2, {n r^(n - 1) {1, I}, r^(n + 1) {(κ0 - n - 1), -I (κ0 + n + 1)}},
  n ≤ -2, {Abs[n] r^(Abs[n] - 1) {1, -I },
   r^(Abs[n] + 1) {(κ0 - Abs[n] - 1), I (κ0 + Abs[n] + 1)}}];
```

Formula for |n|>=2 in fact works for |n|=1, and we therefore redefine soln0:

In[146]:=

```
({n r^(n - 1) {1, I}, r^(n + 1) {(κ0 - n - 1), -I (κ0 + n + 1)}} /. n → 1) -
 {{1, I}, r^2 {(κ0 - 2), -I (κ0 + 2)}}
```

Out[146]=

```
{{0, 0}, {0, 0}}
```

In[148]:=

```
soln0[n_] := Piecewise[{
    {{{r, 0}, {0, r}}, n == 0},
    {{n r^(n - 1) {1, I}, r^(n + 1) {(κ0 - n - 1), -I (κ0 + n + 1)}} Exp[I n ϕ], n ≥ 1},
    {{Abs[n] r^(Abs[n] - 1) {1, -I },
       r^(Abs[n] + 1) {(κ0 - Abs[n] - 1), I (κ0 + Abs[n] + 1)}} Exp[I n ϕ], n ≤ -1}
   }];
```

In[149]:=

```
Table[Map[Lame, soln0[n]], {n, -4, 4}] // Simplify
```

Out[149]=

```
{{{0, 0}, {0, 0}}, {{0, 0}, {0, 0}}, {{0, 0}, {0, 0}},
 {{0, 0}, {0, 0}}, {{0, 0}, {0, 0}}, {{0, 0}, {0, 0}},
 {{0, 0}, {0, 0}}, {{0, 0}, {0, 0}}, {{0, 0}, {0, 0}}}
```

In[150]:=

```
Table[Map[T, soln0[n] ], {n, -2, 2}] // Simplify
```

Out[150]=

$$\left\{\left\{\left\{4\, e^{-2 i \phi}\mu,\ -4\, i\, e^{-2 i \phi}\mu\right\},\ \left\{0,\ 12\, i\, e^{-2 i \phi}\mu\right\}\right\},\ \left\{\{0,0\},\ \left\{4\, e^{-i \phi}\mu,\ 4\, i\, e^{-i \phi}\mu\right\}\right\},\ \{\{2(\lambda+\mu),0\},\{0,0\}\},\ \left\{\{0,0\},\ \left\{4\, e^{i \phi}\mu,\ -4\, i\, e^{i \phi}\mu\right\}\right\},\ \left\{\left\{4\, e^{2 i \phi}\mu,\ 4\, i\, e^{2 i \phi}\mu\right\},\ \left\{0,\ -12\, i\, e^{2 i \phi}\mu\right\}\right\}\right\}$$

In[151]:=

```
Table[Map[TDir, soln0[n]], {n, -2, 2}] // Simplify
```

Out[151]=

$$\left\{\left\{\left\{2\, e^{-2 i \phi} r,\ -2\, i\, e^{-2 i \phi} r\right\},\ \left\{-\frac{2\, e^{-2 i \phi} r^3 \lambda}{\lambda+\mu},\ \frac{2\, i\, e^{-2 i \phi} r^3 (2\lambda+3\mu)}{\lambda+\mu}\right\}\right\},\right.$$
$$\left\{\left\{e^{-i \phi},\ -i\, e^{-i \phi}\right\},\ \left\{-\frac{e^{-i \phi} r^2 (\lambda-\mu)}{\lambda+\mu},\ \frac{i\, e^{-i \phi} r^2 (3\lambda+5\mu)}{\lambda+\mu}\right\}\right\},$$
$$\{\{r,0\},\{0,r\}\},\ \left\{\left\{e^{i \phi},\ i\, e^{i \phi}\right\},\ \left\{-\frac{e^{i \phi} r^2 (\lambda-\mu)}{\lambda+\mu},\ -\frac{i\, e^{i \phi} r^2 (3\lambda+5\mu)}{\lambda+\mu}\right\}\right\},$$
$$\left.\left\{\left\{2\, e^{2 i \phi} r,\ 2\, i\, e^{2 i \phi} r\right\},\ \left\{-\frac{2\, e^{2 i \phi} r^3 \lambda}{\lambda+\mu},\ -\frac{2\, i\, e^{2 i \phi} r^3 (2\lambda+3\mu)}{\lambda+\mu}\right\}\right\}\right\}$$

Solutions of the Lame system for Λ≠0.

We seek these solutions in the form Grad[Φ[*r*, *ϕ*]] + Curl[$\vec{z}$Ψ[r,*ϕ*]]. Then the scalar potentials Φ

and Ψ should satisfy Helmholtz equations, as seen from the calculation below. Indeed, setting ug=Grad[Φ[r, ϕ]] and uc=Curl[$\vec{z}$Ψ[r,ϕ]], we get
Lame[ug] - Λ ug = Grad[-(λ+2μ)ΔΦ - ΛΦ]
and
Lame[uc] - Λ uc = Curl[$\vec{z}$(-μΔΨ - ΛΨ)]

In[152]:=

```
Curl[ Curl[{0, 0, Ψ[r, ϕ]}, {r, ϕ, z}, "Cylindrical"],
  {r, ϕ, z}, "Cylindrical"][[3]]
Laplacian[Ψ[r, ϕ], {r, ϕ, z}, "Cylindrical"]
```

Out[152]=

$$-\frac{\frac{\Psi^{(0,2)}[r,\phi]}{r}+\Psi^{(1,0)}[r,\phi]}{r}-\Psi^{(2,0)}[r,\phi]$$

Out[153]=

$$\frac{\frac{\Psi^{(0,2)}[r,\phi]}{r}+\Psi^{(1,0)}[r,\phi]}{r}+\Psi^{(2,0)}[r,\phi]$$

In[154]:=

```
ug = Grad[Φ[r, ϕ], {r, ϕ, z}, "Cylindrical"][[1 ;; 2]]
eqg1 = FullSimplify[Lame[ug] - Λ ug];
eqg2 = FullSimplify[
 Grad[- (λ + 2 μ) Laplacian[Φ[r, ϕ], {r, ϕ, z}, "Cylindrical"] -
  Λ Φ[r, ϕ], {r, ϕ, z}, "Cylindrical"][[1 ;; 2]]];
eqg1 - eqg2
```

Out[154]=

$$\left\{\Phi^{(1,0)}[r,\phi],\ \frac{\Phi^{(0,1)}[r,\phi]}{r}\right\}$$

Out[157]=

$$\{0,\ 0\}$$

In[158]:=

```
uc = Curl[{0, 0, Ψ[r, ϕ]}, {r, ϕ, z}, "Cylindrical"][[1 ;; 2]]
eqc1 = FullSimplify[Lame[uc] - Λ uc];
eqc2 = FullSimplify[
 Curl[{0, 0, -μ Laplacian[Ψ[r, ϕ], {r, ϕ, z}, "Cylindrical"] - Λ Ψ[r, ϕ]},
  {r, ϕ, z}, "Cylindrical"][[1 ;; 2]]];
eqc1 - eqc2 // Simplify
```

Out[158]=

$$\left\{\frac{\Psi^{(0,1)}[r,\phi]}{r},\ -\Psi^{(1,0)}[r,\phi]\right\}$$

Out[161]=

$$\{0,\ 0\}$$

The solutions of Helmholtz equation -ΔΘ-kΘ=0 are
$\theta = J_{|n|}\left(\sqrt{k}\, r\right)\exp(I\, n\, \phi)$ for $k>0$ and $\Theta = I_{|n|}\left(\sqrt{-k}\ r\right)\ \exp(I\ n\ \phi)$ for $k<0$.
We will only deal with non-negative n's remembering later to double the multiplicities for n≠0.
solnneg refers to k<0 and may be ignored

In[162]:=

```
solnpos[n_] :=
 {Grad[BesselJ[n, Sqrt[Λ / (λ + 2 μ)] r] Exp[I n ϕ], {r, ϕ, z}, "Cylindrical"][[1 ;; 2]],
   Curl[{0, 0, BesselJ[n, Sqrt[Λ / μ] r] Exp[I n ϕ]}, {r, ϕ, z}, "Cylindrical"][[
    1 ;; 2]]} /. BesselJ[-1 + n, z_] → (2 n BesselJ[n, z] / z - BesselJ[n + 1, z])
solnneg[n_] :=
  {Grad[BesselI[n, Sqrt[-Λ / (λ + 2 μ)] r] Exp[I n ϕ], {r, ϕ, z}, "Cylindrical"][[
      1 ;; 2]], Curl[{0, 0, BesselI[n, Sqrt[-Λ / μ] r] Exp[I n ϕ]},
       {r, ϕ, z}, "Cylindrical"][[1 ;; 2]]} /.
    BesselI[-1 + n, z_] → (2 n BesselI[n, z] / z + BesselI[n + 1, z]) // FullSimplify;
```

Just to check that we indeed produce solutions of Lame system for both positive and negative n:

In[164]:=

```
Map[(Lame[#] - Λ #) &, solnpos[n]] // FullSimplify
```

Out[164]=

```
{{0, 0}, {0, 0}}
```

In[165]:=

```
solnposMinusn[n_] :=
  {Grad[BesselJ[n, Sqrt[Λ / (λ + 2 μ)] r] Exp[-I n ϕ], {r, ϕ, z}, "Cylindrical"][[
     1 ;; 2]], Curl[{0, 0, BesselJ[n, Sqrt[Λ / μ] r] Exp[-I n ϕ]},
      {r, ϕ, z}, "Cylindrical"][[1 ;; 2]]} /.
   BesselJ[-1 + n, z_] → (2 n BesselJ[n, z] / z - BesselJ[n + 1, z]);
```

In[166]:=

```
Map[(Lame[#] - Λ #) &, solnposMinusn[n]] // FullSimplify
```

Out[166]=

```
{{0, 0}, {0, 0}}
```

This is to check that choosing plus or minus n does not affect b.c. : the first (gradient) solutions changes the radial part as (u,v)->(u,-v) under n->-n,
the second (curl) solution changes the radial part as (u,v)->(-u,v) under n->-n, so homogeneous conditions stay the same. So from now on we just deal with n>= 0 and double all the eigenvalues for non-zero n

In[167]:=

```
Exp[-I n ϕ] solnpos[n]⟦1⟧ -
  Exp[I n ϕ] {{1, 0}, {0, -1}} .solnposMinusn[n]⟦1⟧ // Simplify
Exp[-I n ϕ] solnpos[n]⟦2⟧ -
  Exp[I n ϕ] {{-1, 0}, {0, 1}} .solnposMinusn[n]⟦2⟧ // Simplify
Exp[-I n ϕ] T[solnpos[n]⟦1⟧] -
  Exp[I n ϕ] {{1, 0}, {0, -1}} .T[solnposMinusn[n]⟦1⟧] // Simplify
Exp[-I n ϕ] T[solnpos[n]⟦2⟧] -
  Exp[I n ϕ] {{-1, 0}, {0, 1}} .T[solnposMinusn[n]⟦2⟧] // Simplify
```

Out[167]=

{0, 0}

Out[168]=

{0, 0}

Out[169]=

{0, 0}

Out[170]=

{0, 0}

The Dirichlet eigenvalues are the Λ-roots of

In[171]:=

```
(Map[(# /. {r → 1, Exp[_] → 1}) &, solnpos[n]] // Det) // FullSimplify
```

Out[171]=

$$n \sqrt{\frac{\Lambda}{\mu}}\, \text{BesselJ}\left[n, \sqrt{\frac{\Lambda}{\lambda + 2\mu}}\right] \text{BesselJ}\left[1+n, \sqrt{\frac{\Lambda}{\mu}}\right] + \sqrt{\frac{\Lambda}{\lambda + 2\mu}} \left(n\, \text{BesselJ}\left[n, \sqrt{\frac{\Lambda}{\mu}}\right] - \sqrt{\frac{\Lambda}{\mu}}\, \text{BesselJ}\left[1+n, \sqrt{\frac{\Lambda}{\mu}}\right]\right) \text{BesselJ}\left[1+n, \sqrt{\frac{\Lambda}{\lambda + 2\mu}}\right]$$

In[172]:=

```
Direigseq = (Map[(# /. {r → 1, Exp[_] → 1}) &, solnpos[n]] // Det) /.
   BesselJ[-1 + n, z_] → (2 n BesselJ[n, z] / z - BesselJ[n + 1, z]) // FullSimplify
```

Out[172]=

$$n \sqrt{\frac{\Lambda}{\mu}}\, \text{BesselJ}\left[n, \sqrt{\frac{\Lambda}{\lambda + 2\mu}}\right] \text{BesselJ}\left[1+n, \sqrt{\frac{\Lambda}{\mu}}\right] + \sqrt{\frac{\Lambda}{\lambda + 2\mu}} \left(n\, \text{BesselJ}\left[n, \sqrt{\frac{\Lambda}{\mu}}\right] - \sqrt{\frac{\Lambda}{\mu}}\, \text{BesselJ}\left[1+n, \sqrt{\frac{\Lambda}{\mu}}\right]\right) \text{BesselJ}\left[1+n, \sqrt{\frac{\Lambda}{\lambda + 2\mu}}\right]$$

In[183]:=

```
Direigseqω = (Direigseq /. {Λ/(λ + 2 μ) → ω1,Λ, Λ/μ → ω2,Λ, n → k})
Direigseqω /. n → 0
Direigseqω // TeXForm
```

Out[183]=

$$\text{BesselJ}\left[1+k,\sqrt{\omega_{1,\Lambda}}\right]\sqrt{\omega_{1,\Lambda}}\left(k\,\text{BesselJ}\left[k,\sqrt{\omega_{2,\Lambda}}\right]-\text{BesselJ}\left[1+k,\sqrt{\omega_{2,\Lambda}}\right]\sqrt{\omega_{2,\Lambda}}\right)+k\,\text{BesselJ}\left[k,\sqrt{\omega_{1,\Lambda}}\right]\text{BesselJ}\left[1+k,\sqrt{\omega_{2,\Lambda}}\right]\sqrt{\omega_{2,\Lambda}}$$

Out[184]=

$$\text{BesselJ}\left[1+k,\sqrt{\omega_{1,\Lambda}}\right]\sqrt{\omega_{1,\Lambda}}\left(k\,\text{BesselJ}\left[k,\sqrt{\omega_{2,\Lambda}}\right]-\text{BesselJ}\left[1+k,\sqrt{\omega_{2,\Lambda}}\right]\sqrt{\omega_{2,\Lambda}}\right)+k\,\text{BesselJ}\left[k,\sqrt{\omega_{1,\Lambda}}\right]\text{BesselJ}\left[1+k,\sqrt{\omega_{2,\Lambda}}\right]\sqrt{\omega_{2,\Lambda}}$$

Out[185]//TeXForm=

```
k \sqrt{\omega _{2,\Lambda }} J_k\left(\sqrt{\omega _{1,\Lambda }}\right)
   J_{k+1}\left(\sqrt{\omega _{2,\Lambda }}\right)+\sqrt{\omega _{1,\Lambda }}
   J_{k+1}\left(\sqrt{\omega _{1,\Lambda }}\right) \left(k J_k\left(\sqrt{\omega
   _{2,\Lambda }}\right)-\sqrt{\omega _{2,\Lambda }} J_{k+1}\left(\sqrt{\omega
   _{2,\Lambda }}\right)\right)
```

and the Neumann eigenvalues are Λ-roots of (this sometimes does not show as Bessels but as Hypergeometric functions)

In[186]:=

```
Neueigseq = (Map[(T[#] /. {r → 1, Exp[_] → 1}) &, solnpos[n]] // Det) /.
   BesselJ[-1 + n, z_] → (2 n BesselJ[n, z] / z - BesselJ[n + 1, z]) // FullSimplify
```

Out[186]=

$$2\sqrt{\frac{\Lambda}{\mu}}\,\mu\,\text{BesselJ}\left[1+n,\sqrt{\frac{\Lambda}{\mu}}\right]\left(\left(\Lambda+2n\mu-2n^3\mu\right)\text{BesselJ}\left[n,\sqrt{\frac{\Lambda}{\lambda+2\mu}}\right]+2\left(-1+n^2\right)\mu\sqrt{\frac{\Lambda}{\lambda+2\mu}}\,\text{BesselJ}\left[1+n,\sqrt{\frac{\Lambda}{\lambda+2\mu}}\right]\right)+\text{BesselJ}\left[n,\sqrt{\frac{\Lambda}{\mu}}\right]\left(-\Lambda\left(\Lambda-4\left(-1+n\right)n\mu\right)\text{BesselJ}\left[n,\sqrt{\frac{\Lambda}{\lambda+2\mu}}\right]+2\mu\sqrt{\frac{\Lambda}{\lambda+2\mu}}\left(\Lambda+2n\mu-2n^3\mu\right)\text{BesselJ}\left[1+n,\sqrt{\frac{\Lambda}{\lambda+2\mu}}\right]\right)$$

In[192]:=

```
Direq[λ_, μ_, n_, Λ_] :=
  n Sqrt[1/μ] BesselJ[n, Sqrt[Λ/(λ + 2 μ)]] BesselJ[1 + n, Sqrt[Λ/μ]] + Sqrt[1/(λ + 2 μ)]
   (n BesselJ[n, Sqrt[Λ/μ]] - Sqrt[Λ/μ] BesselJ[1 + n, Sqrt[Λ/μ]]) BesselJ[1 + n, Sqrt[Λ/(λ + 2 μ)]];

Simplify[Direq[λ, μ, n, Λ] - Direigseq / Sqrt[Λ], Λ > 0 && μ > 0 && λ + 2 μ > 0]
```

Out[196]=

0

In[197]:=

```
Direq[λ, μ, n, Λ] // FullSimplify
Direq[λ, μ, 0, Λ] // Simplify
```

Out[197]=

$$n \sqrt{\frac{1}{\mu}}\ \text{BesselJ}\left[n, \sqrt{\frac{\Lambda}{\lambda+2\mu}}\right] \text{BesselJ}\left[1+n, \sqrt{\frac{\Lambda}{\mu}}\right] + \sqrt{\frac{1}{\lambda+2\mu}} \left(n\,\text{BesselJ}\left[n, \sqrt{\frac{\Lambda}{\mu}}\right] - \sqrt{\frac{\Lambda}{\mu}}\ \text{BesselJ}\left[1+n, \sqrt{\frac{\Lambda}{\mu}}\right]\right) \text{BesselJ}\left[1+n, \sqrt{\frac{\Lambda}{\lambda+2\mu}}\right]$$

Out[198]=

$$-\sqrt{\frac{\Lambda}{\mu}} \sqrt{\frac{1}{\lambda+2\mu}}\ \text{BesselJ}\left[1, \sqrt{\frac{\Lambda}{\mu}}\right] \text{BesselJ}\left[1, \sqrt{\frac{\Lambda}{\lambda+2\mu}}\right]$$

Neueq[λ_, μ_, n_, Λ_] :=

$$2\sqrt{\frac{\Lambda}{\mu}}\ \mu\,\text{BesselJ}\left[1+n, \sqrt{\frac{\Lambda}{\mu}}\right] \left(\left(\Lambda + 2 n \mu - 2 n^3 \mu\right) \text{BesselJ}\left[n, \sqrt{\frac{\Lambda}{\lambda+2\mu}}\right] + 2\left(-1+n^2\right)\mu \sqrt{\frac{\Lambda}{\lambda+2\mu}}\ \text{BesselJ}\left[1+n, \sqrt{\frac{\Lambda}{\lambda+2\mu}}\right]\right) + \text{BesselJ}\left[n, \sqrt{\frac{\Lambda}{\mu}}\right] \left(-\Lambda\,(\Lambda - 4\,(-1+n)\,n\,\mu)\ \text{BesselJ}\left[n, \sqrt{\frac{\Lambda}{\lambda+2\mu}}\right] + 2\mu \sqrt{\frac{\Lambda}{\lambda+2\mu}} \left(\Lambda + 2 n \mu - 2 n^3 \mu\right) \text{BesselJ}\left[1+n, \sqrt{\frac{\Lambda}{\lambda+2\mu}}\right]\right);$$

```
Simplify[Neueq[λ, μ, n, Λ] - Neueigseq, Λ > 0 && μ > 0 && λ + 2 μ > 0]
```

Out[200]=

0

In[201]:=

```
Direigs[λ0_, μ0_, nmax_, Λmax_] := Module[{t, j},
   t = Table[findAllRoots[Direq[λ0, μ0, n, Λ], {Λ, 0., Λmax}], {n, 0, nmax}];
   t = {t⟦1⟧, Table[{t⟦j⟧, t⟦j⟧}, {j, 2, Length[t]}]} // Flatten;
   t // Sort
  ];
DireigsWithn[λ0_, μ0_, nmax_, Λmax_] := Module[{t, t1, t2, t3, pair},
   t = Table[findAllRoots[Direq[λ0, μ0, n, Λ], {Λ, 0., Λmax}], {n, 0, nmax}];
   t1 = Select[Table[If[Length[t⟦n + 1⟧] > 0, Thread[{n, t⟦n + 1⟧}]],
      {n, 0, nmax}], Length[#] > 0 &];
   t2 = SortBy[Flatten[t1, 1], N[#⟦2⟧] &];
   t2
  ];
Neueigs[λ0_, μ0_, nmax_, Λmax_] := Module[{t, k},
  t = Table[findAllRoots[Neueq[λ0, μ0, n, Λ], {Λ, 0., Λmax}], {n, 0, nmax}];
  t = {t⟦1⟧, Table[{t⟦j⟧, t⟦j⟧}, {j, 2, Length[t]}]} // Flatten;
  {0., 0., 0., t // Sort} // Flatten
 ]
NeueigsWithn[λ0_, μ0_, nmax_, Λmax_] := Module[{t, t1, t2, t3, pair},
   t = Table[findAllRoots[Neueq[λ0, μ0, n, Λ], {Λ, 0., Λmax}], {n, 0, nmax}];
   t1 = Select[Table[If[Length[t⟦n + 1⟧] > 0, Thread[{n, t⟦n + 1⟧}]],
      {n, 0, nmax}], Length[#] > 0 &];
   t2 = SortBy[Flatten[t1, 1], #⟦2⟧ &];
   PrependTo[t2, {1, 0.}];
   PrependTo[t2, {0, 0.}];
   t2
  ];
```

Examples:

In[205]:=

```
Direigs[0, 1, 52, 3200.]
DireigsWithn[0, 1, 52, 3200.]
```

Out[205]=

```
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Out[206]=

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In[209]:=

```
MyNlambdaDisk[evs_, Λ_] :=
  Sum[If[evs⟦j⟧⟦1⟧ == 0, 1, 2] UnitStep[Λ - evs⟦j⟧⟦2⟧], {j, 1, Length[evs]}];
```

Finding the number of harmonics to stop at

In[210]:=

```
findAllRoots[Direq[-0.5, 1, 47, Λ], {Λ, 0., 3000}]
findAllRoots[Direq[-0.5, 1, 48, Λ], {Λ, 0., 3000}]
```

Out[210]=

```
{2961.62}
```

Out[211]=

```
{}
```

In[212]:=

```
findAllRoots[Direq[3, 1, 46, Λ], {Λ, 0., 3000}]
findAllRoots[Direq[3, 1, 47, Λ], {Λ, 0., 3000}]
```

Out[212]=

```
{2900.41}
```

Out[213]=

```
{}
```

In[214]:=

```
findAllRoots[Direq[100, 1, 46, Λ], {Λ, 0., 3000}]
findAllRoots[Direq[100, 1, 47, Λ], {Λ, 0., 3000}]
```

Out[214]=

```
{2913.57}
```

Out[215]=

```
{}
```

Dirichlet counting functions

In[216]:=

```
evD1 = DireigsWithn[-0.5, 1, 47, 3000];
evD2 = DireigsWithn[3, 1, 46, 3000];
evD3 = DireigsWithn[100, 1, 46, 3000];
```

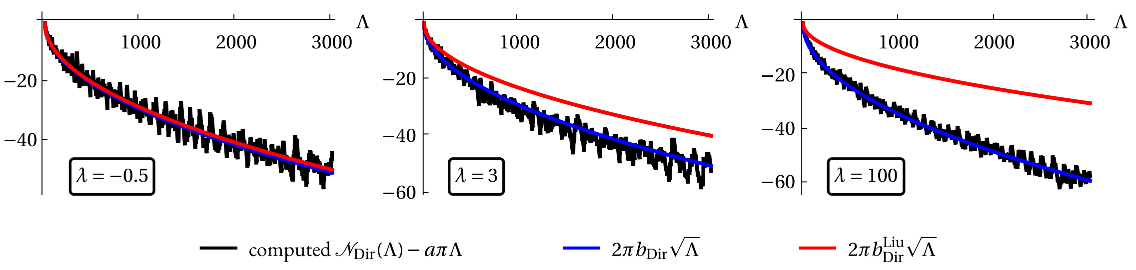
In[219]:=

```
xtdisk = Table[{x, MaTeX[x]}, {x, 1000, 3000, 1000}];
ytdisk = Table[{x, MaTeX[x]}, {x, -60, -20, 20}];
```

In[221]:=

```
figdisk1 = Plot[{MyNlambdaDisk[evD1, Λ] - CWeyl[2, -0.5, 1] Pi Λ,
    2 Pi BDir[2, αλ[-0.5, 1], 1] Sqrt[Λ], 2 Pi BDirLiu[2, -0.5, 1] Sqrt[Λ]},
   {Λ, 0, 3000}, PlotStyle → {Black, Blue, Red}, Exclusions → None,
   Ticks → {xtdisk, ytdisk}, AxesLabel → {MaTeX["\\Lambda"], None}, Epilog →
    Inset[Framed[MaTeX["\\lambda=-0.5"], RoundingRadius → 2, ContentPadding →
       False, FrameMargins → Tiny], Scaled[{0.1, 0}], Scaled[{0, 0}]]];
figdisk2 = Plot[{MyNlambdaDisk[evD2, Λ] - CWeyl[2, 3, 1] Pi Λ,
    2 Pi BDir[2, αλ[3, 1], 1] Sqrt[Λ], 2 Pi BDirLiu[2, 3, 1] Sqrt[Λ]},
   {Λ, 0, 3000}, PlotStyle → {Black, Blue, Red}, Exclusions → None,
   Ticks → {xtdisk, ytdisk}, AxesLabel → {MaTeX["\\Lambda"], None}, Epilog →
    Inset[Framed[MaTeX["\\lambda=3"], RoundingRadius → 2, ContentPadding →
       False, FrameMargins → Tiny], Scaled[{0.1, 0}], Scaled[{0, 0}]]];
figdisk3 = Plot[{MyNlambdaDisk[evD3, Λ] - CWeyl[2, 100, 1] Pi Λ,
    2 Pi BDir[2, αλ[100, 1], 1] Sqrt[Λ], 2 Pi BDirLiu[2, 100, 1] Sqrt[Λ]},
   {Λ, 0, 3000}, PlotStyle → {Black, Blue, Red}, Exclusions → None,
   Ticks → {xtdisk, ytdisk}, AxesLabel → {MaTeX["\\Lambda"], None}, Epilog →
    Inset[Framed[MaTeX["\\lambda=100"], RoundingRadius → 2, ContentPadding →
       False, FrameMargins → Tiny], Scaled[{0.1, 0}], Scaled[{0, 0}]]] ;
legendisk = LineLegend[{Black, Blue, Red},
   MaTeX[{"\\text{computed }\\mathcal{N}_{\\mathrm{Dir}}(\\Lambda)-a\\pi
       \\Lambda\\qquad\\qquad",
     "2\\pi b_{\\mathrm{Dir}}\\sqrt{\\Lambda}\\qquad\\qquad",
     "2\\pi b_{\\mathrm{Dir}}^{\\mathrm{Liu}}\\sqrt{\\Lambda}"}],
   LegendFunction → None, LegendLayout → {"Row", 1}];
figdisk = GraphicsColumn[{GraphicsRow[{figdisk1, figdisk2, figdisk3},
    ImageSize → Full, Spacings → {Scaled[0.05], Automatic}],
   legendisk}, ImageSize → Full, Spacings → {0, Automatic}]
```

Out[225]=



In[ ]:=

```
Export["figdisk.pdf", figdisk]
```

Out[ ]=

```
figdisk.pdf
```

Finding the number of harmonics to stop at

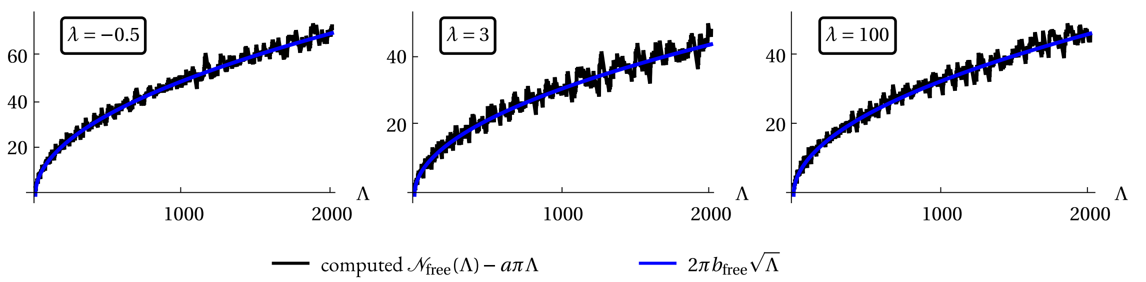
In[226]:=
```
findAllRoots[Neueq[-0.5, 1, 56, Λ], {Λ, 0., 2000}]
findAllRoots[Neueq[-0.5, 1, 57, Λ], {Λ, 0., 2000}]
```
Out[226]=
```
{1992.}
```
Out[227]=
```
{}
```
In[228]:=
```
findAllRoots[Neueq[3, 1, 45, Λ], {Λ, 0., 2000}]
findAllRoots[Neueq[3, 1, 46, Λ], {Λ, 0., 2000}]
```
Out[228]=
```
{1929.28}
```
Out[229]=
```
{}
```
In[230]:=
```
findAllRoots[Neueq[100, 1, 44, Λ], {Λ, 0., 2000}]
findAllRoots[Neueq[100, 1, 45, Λ], {Λ, 0., 2000}]
```
Out[230]=
```
{1915.88}
```
Out[231]=
```
{}
```

In[232]:=
```
evDfr1 = NeueigsWithn[-0.5, 1, 56, 2000];
evDfr2 = NeueigsWithn[3, 1, 45, 2000];
evDfr3 = NeueigsWithn[100, 1, 44, 2000];
```
In[235]:=
```
ytfrdisk = Table[{x, MaTeX["\\ " <> ToString[x]]}, {x, 20, 60, 20}];
```

In[236]:=

```
figdiskfr1 = Plot[{MyNlambdaDisk[evDfr1, Λ] - CWeyl[2, -0.5, 1] Pi Λ,
    2 Pi Bfree[2, αλ[-0.5, 1], 1] Sqrt[Λ]}, {Λ, 0, 2000},
   PlotStyle → {Black, Blue}, Exclusions → None, Ticks → {xtdisk, ytfrdisk},
   AxesLabel → {MaTeX["\\Lambda"], None}, Epilog → Inset[
     Framed[MaTeX["\\lambda=-0.5"], RoundingRadius → 2, ContentPadding → False,
      FrameMargins → Tiny], Scaled[{0.1, 1}], Scaled[{0, 1}]]];
figdiskfr2 = Plot[{MyNlambdaDisk[evDfr2, Λ] - CWeyl[2, 3, 1] Pi Λ,
    2 Pi Bfree[2, αλ[3, 1], 1] Sqrt[Λ]}, {Λ, 0, 2000},
   PlotStyle → {Black, Blue}, Exclusions → None, Ticks → {xtdisk, ytfrdisk},
   AxesLabel → {MaTeX["\\Lambda"], None}, Epilog → Inset[
     Framed[MaTeX["\\lambda=3"], RoundingRadius → 2, ContentPadding → False,
      FrameMargins → Tiny], Scaled[{0.1, 1}], Scaled[{0, 1}]]];
figdiskfr3 = Plot[{MyNlambdaDisk[evDfr3, Λ] - CWeyl[2, 100, 1] Pi Λ,
    2 Pi Bfree[2, αλ[100, 1], 1] Sqrt[Λ]}, {Λ, 0, 2000},
   PlotStyle → {Black, Blue}, Exclusions → None, Ticks → {xtdisk, ytfrdisk},
   AxesLabel → {MaTeX["\\Lambda"], None}, Epilog → Inset[
     Framed[MaTeX["\\lambda=100"], RoundingRadius → 2, ContentPadding → False,
      FrameMargins → Tiny], Scaled[{0.1, 1}], Scaled[{0, 1}]]] ;
legendiskfr = LineLegend[{Black, Blue},
   MaTeX[{"\\text{computed }\\mathcal{N}_{\\mathrm{free}}(\\Lambda)-a\\pi
       \\Lambda\\qquad\\qquad",
     "2\\pi b_{\\mathrm{free}}\\sqrt{\\Lambda}\\qquad\\qquad"}],
   LegendFunction → None, LegendLayout → {"Row", 1}];
figdiskfr = GraphicsColumn[{GraphicsRow[{figdiskfr1, figdiskfr2, figdiskfr3},
    ImageSize → Full, Spacings → {Scaled[0.05], Automatic}],
   legendiskfr}, ImageSize → Full, Spacings → {0, Automatic}]
```

Out[240]=



In[ ]:=

```
Export["figdiskfree.pdf", figdiskfr]
```

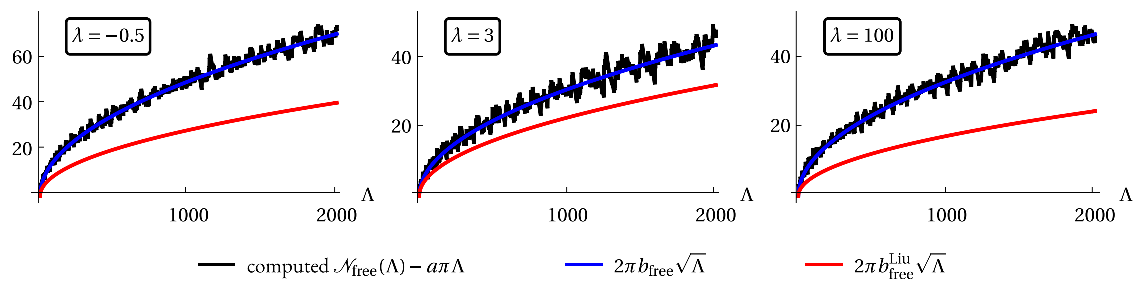
Out[ ]=

figdiskfree.pdf

In[241]:=

```
figdiskfr1new = Plot[{MyNlambdaDisk[evDfr1, Λ] - CWeyl[2, -0.5, 1] Pi Λ,
    2 Pi Bfree[2, αλ[-0.5, 1], 1] Sqrt[Λ], -2 Pi BDirLiu[2, -0.5, 1] Sqrt[Λ]},
   {Λ, 0, 2000}, PlotStyle → {Black, Blue, Red}, Exclusions → None,
   Ticks → {xtdisk, ytfrdisk}, AxesLabel → {MaTeX["\\Lambda"], None}, Epilog →
    Inset[Framed[MaTeX["\\lambda=-0.5"], RoundingRadius → 2, ContentPadding →
       False, FrameMargins → Tiny], Scaled[{0.1, 1}], Scaled[{0, 1}]]];
figdiskfr2new = Plot[{MyNlambdaDisk[evDfr2, Λ] - CWeyl[2, 3, 1] Pi Λ,
    2 Pi Bfree[2, αλ[3, 1], 1] Sqrt[Λ], -2 Pi BDirLiu[2, 3, 1] Sqrt[Λ]},
   {Λ, 0, 2000}, PlotStyle → {Black, Blue, Red}, Exclusions → None,
   Ticks → {xtdisk, ytfrdisk}, AxesLabel → {MaTeX["\\Lambda"], None}, Epilog →
    Inset[Framed[MaTeX["\\lambda=3"], RoundingRadius → 2, ContentPadding →
       False, FrameMargins → Tiny], Scaled[{0.1, 1}], Scaled[{0, 1}]]];
figdiskfr3new = Plot[{MyNlambdaDisk[evDfr3, Λ] - CWeyl[2, 100, 1] Pi Λ,
    2 Pi Bfree[2, αλ[100, 1], 1] Sqrt[Λ], -2 Pi BDirLiu[2, 100, 1] Sqrt[Λ]},
   {Λ, 0, 2000}, PlotStyle → {Black, Blue, Red}, Exclusions → None,
   Ticks → {xtdisk, ytfrdisk}, AxesLabel → {MaTeX["\\Lambda"], None}, Epilog →
    Inset[Framed[MaTeX["\\lambda=100"], RoundingRadius → 2, ContentPadding →
       False, FrameMargins → Tiny], Scaled[{0.1, 1}], Scaled[{0, 1}]]] ;
legendiskfrnew = LineLegend[{Black, Blue, Red},
   MaTeX[{"\\text{computed }\\mathcal{N}_{\\mathrm{free}}(\\Lambda)-a\\pi
       \\Lambda\\qquad\\qquad",
     "2\\pi b_{\\mathrm{free}}\\sqrt{\\Lambda}\\qquad\\qquad",
     "2\\pi b_{\\mathrm{free}}^{\\mathrm{Liu}}\\sqrt{\\Lambda}"}],
   LegendFunction → None, LegendLayout → {"Row", 1}];
figdiskfrnew =
 GraphicsColumn[{GraphicsRow[{figdiskfr1new, figdiskfr2new, figdiskfr3new},
    ImageSize → Full, Spacings → {Scaled[0.05], Automatic}],
   legendiskfrnew}, ImageSize → Full, Spacings → {0, Automatic}]
```

Out[245]=



In[ ]:=

```
Export["figdiskfreenew.pdf", figdiskfrnew]
```

Out[ ]=

```
figdiskfreenew.pdf
```