```mathematica
SetDirectory[NotebookDirectory[]];
<< MaTeX`
texStyle = {};
SetOptions[MaTeX,
  "BasePreamble" → {"\\usepackage{amsmath}", "\\usepackage{xcolor}",
    "\\usepackage{fourier}", "\\usepackage{ebgaramond}"}, FontSize → 11];
```

```mathematica
frame[legend_] :=
  Framed[legend, FrameStyle → Thin, RoundingRadius → 10, FrameMargins → 0];
```

```mathematica
αλ[λ_, μ_] := μ / (λ + 2 μ);
λα[α_, μ_] := μ (1 / α - 2);
CWeyl[d_, λ_, μ_] :=
  ((λ + 2 μ) ^ (-d / 2) + (d - 1) μ ^ (-d / 2)) / (4 Pi) ^ (d / 2) / Gamma[1 + d / 2];
BDirLiu[d_, λ_, μ_] :=
  - ((d - 1) / μ ^ ((d - 1) / 2) + 1 / (λ + 2 μ) ^ ((d - 1) / 2)) / 4 / (4 Pi) ^ ((d - 1) / 2) /
    Gamma[1 + (d - 1) / 2];
BDir[d_?NumericQ, α_?NumericQ, μ_?NumericQ] :=
  -μ ^ ((1 - d) / 2) / (2 ^ (d + 1) Pi ^ ((d - 1) / 2) Gamma[(1 + d) / 2])
   (4 (d - 1) / Pi NIntegrate[τ ^ (d - 2) ArcTan[Sqrt[(1 - α τ ^ (-2)) (τ ^ (-2) - 1)]],
       {τ, Sqrt[α], 1}] + α ^ ((d - 1) / 2) + d - 1);
BDirSaVa2[α_, μ_] := 1 / (4 Pi Sqrt[μ]) (-1 - Sqrt[α] -
     4 / Pi NIntegrate[ArcTan[Sqrt[(1 - α / ξ^2) (1 / ξ^2 - 1)]], {ξ, Sqrt[α], 1}]);
BDirSaVa3[λ_, μ_] := -1 / (16 Pi) (3 λ^2 + 13 λ μ + 16 μ^2) / (λ^2 μ + 5 λ μ^2 + 6 μ^3);
γR[α_] := Sqrt[Min[x /. Solve[x^3 - 8 x^2 + 16 (-1 + α) + 8 (3 - 2 α) x == 0, x, Reals]]];
Bfree[d_, α_, μ_] := μ ^ ((1 - d) / 2) / (2 ^ (d + 1) Pi ^ ((d - 1) / 2) Gamma[(1 + d) / 2])
    (4 (d - 1) / Pi NIntegrate[
       τ ^ (d - 2) ArcTan[(τ ^ (-2) - 2) ^2 / (4 Sqrt[(1 - α τ ^ (-2)) (τ ^ (-2) - 1)])],
       {τ, Sqrt[α], 1}] + α ^ ((d - 1) / 2) + d - 5 + 4 γR[α] ^ (1 - d));
BfreeSaVa2[α_, μ_] :=
  1 / (4 Pi Sqrt[μ]) (4 / γR[α] - 3 + Sqrt[α] + 4 / Pi NIntegrate[ArcTan[
        (2 - 1 / ξ^2) ^2 / (4 Sqrt[(1 - α / ξ^2) (1 / ξ^2 - 1)])]], {ξ, Sqrt[α], 1}]);
BfreeSaVa3[λ_, μ_] :=
  1 / (16 Pi) (3 (λ + 2 μ) ^2 - 3 (λ + 2 μ) μ + 2 μ^2) / ((λ + 2 μ) μ (λ + μ));
BDirOdd[k_, α_, μ_] := -μ^k / (2 ^ (2 k + 2) Pi^k (k!))
   (2 / (k!) (D[(2 t - 1 - 1 / α) / (t - 1 - 1 / α) / Sqrt[(1 - α t) (1 - t)], {t, k}] /.
       t → 0) - 2 (α / (α + 1)) ^k + α^k + 2 k);
BfreeOdd[k_, α_, μ_] :=
  μ^k / (2 ^ (2 k + 2) Pi^k (k!)) (-8 / (k!) (D[(t - 2) (2 α t^2 + (α - 3) t + 2 (1 - α)) /
          (t^3 - 8 t^2 + 8 (3 - 2 α) t + 16 (α - 1)) ×
          (1 / Sqrt[(1 - α t) (1 - t)] + 4 / (t - 2) ^2), {t, k}] /. t → 0) -
     α^k + 2 (k + 2 ^ (2 - k) - 1));
```

In[449]:=

```
tabDirodd = Table[BDirOdd[k, α, μ] // Simplify, {k, 1, 4}] // TableForm
tabDirodd // TeXForm
```

Out[449]//TableForm=

$$-\frac{\left(3+\alpha+2\,\alpha^2\right)\mu}{16\,\pi\,(1+\alpha)}$$

$$-\frac{\left(19+36\,\alpha+6\,\alpha^2+12\,\alpha^3+7\,\alpha^4\right)\mu^2}{512\,\pi^2\,(1+\alpha)^2}$$

$$-\frac{\left(53+156\,\alpha+147\,\alpha^2+16\,\alpha^3+27\,\alpha^4+36\,\alpha^5+13\,\alpha^6\right)\mu^3}{12\,288\,\pi^3\,(1+\alpha)^3}$$

$$-\frac{\left(547+2168\,\alpha+3188\,\alpha^2+1992\,\alpha^3+146\,\alpha^4+200\,\alpha^5+500\,\alpha^6+376\,\alpha^7+99\,\alpha^8\right)\mu^4}{1\,572\,864\,\pi^4\,(1+\alpha)^4}$$

Out[450]//TeXForm=

```
\begin{array}{c}
 -\frac{\left(2 \alpha ^2+\alpha +3\right) \mu }{16 \pi  (\alpha +1)} \\
 -\frac{\left(7 \alpha ^4+12 \alpha ^3+6 \alpha ^2+36 \alpha +19\right) \mu ^2}{512 \pi
   ^2 (\alpha +1)^2} \\
 -\frac{\left(13 \alpha ^6+36 \alpha ^5+27 \alpha ^4+16 \alpha ^3+147 \alpha ^2+156
   \alpha +53\right) \mu ^3}{12288 \pi ^3 (\alpha +1)^3} \\
 -\frac{\left(99 \alpha ^8+376 \alpha ^7+500 \alpha ^6+200 \alpha ^5+146 \alpha ^4+1992
   \alpha ^3+3188 \alpha ^2+2168 \alpha +547\right) \mu ^4}{1572864 \pi ^4 (\alpha +1)^4
   } \\
\end{array}
```

In[459]:=

```
tabfreeodd = Table[Simplify[BfreeOdd[k, α, μ], α < 1], {k, 1, 4}] // TableForm
tabfreeodd // TeXForm
```

Out[459]//TableForm=

$$-\frac{\left(3-3\,\alpha+2\,\alpha^2\right)\mu}{16\,\pi\,(-1+\alpha)}$$

$$-\frac{\left(-21+36\,\alpha-14\,\alpha^2-12\,\alpha^3+7\,\alpha^4\right)\mu^2}{512\,\pi^2\,(-1+\alpha)^2}$$

$$-\frac{\left(62-168\,\alpha+159\,\alpha^2-56\,\alpha^3+30\,\alpha^4-36\,\alpha^5+13\,\alpha^6\right)\mu^3}{12\,288\,\pi^3\,(-1+\alpha)^3}$$

$$-\frac{\left(-661+2440\,\alpha-3468\,\alpha^2+2200\,\alpha^3-470\,\alpha^4-296\,\alpha^5+516\,\alpha^6-376\,\alpha^7+99\,\alpha^8\right)\mu^4}{1\,572\,864\,\pi^4\,(-1+\alpha)^4}$$

Out[460]//TeXForm=

```
\begin{array}{c}
 -\frac{\left(2 \alpha ^2-3 \alpha +3\right) \mu }{16 \pi  (\alpha -1)} \\
 -\frac{\left(7 \alpha ^4-12 \alpha ^3-14 \alpha ^2+36 \alpha -21\right) \mu ^2}{512 \pi
   ^2 (\alpha -1)^2} \\
 -\frac{\left(13 \alpha ^6-36 \alpha ^5+30 \alpha ^4-56 \alpha ^3+159 \alpha ^2-168
   \alpha +62\right) \mu ^3}{12288 \pi ^3 (\alpha -1)^3} \\
 -\frac{\left(99 \alpha ^8-376 \alpha ^7+516 \alpha ^6-296 \alpha ^5-470 \alpha ^4+2200
   \alpha ^3-3468 \alpha ^2+2440 \alpha -661\right) \mu ^4}{1572864 \pi ^4 (\alpha -1)^4
   } \\
\end{array}
```

In[461]:=

```
xt = Table[{x, MaTeX[x]}, {x, 0.2, 1, 0.2}];
yt = Table[{y, MaTeX[y]}, {y, 0.5, 1, 0.1}];
xtb = Table[{x, MaTeX[x]}, {x, 0.7, 1, 0.1}];
ytb = Table[{y, MaTeX[y]}, {y, 0.98, 1, 0.01}];
```

```
Off[NIntegrate::nlim]; figbLtobi =
 Plot[Table[BDirLiu[d, λα[α, 1], 1] / BDir[d, α, 1], {d, {2, 3, 4, 5}}] // Evaluate,
  {α, 0, 1}, PlotRange → {0.4, 1}, Ticks → {xt, yt}, AxesLabel →
   MaTeX[{"\\alpha", "b^\\mathrm{Liu}_\\mathrm{Dir}/b_\\mathrm{Dir}"}],
  PlotLegends → LineLegend[97, MaTeX[{"d=2", "d=3", "d=4", "d=5"}],
   LegendFunction → None, LegendLayout → {"Column", 1}]];
(* ignore warnings! *)
```

⋯ NIntegrate: $\tau = \sqrt{\alpha}$ is not a valid limit of integration. ⓘ

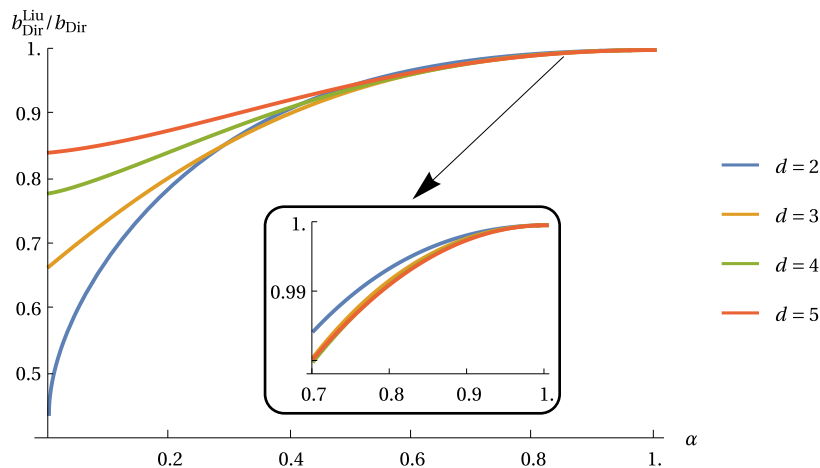⋯ NIntegrate: $\tau = \sqrt{\alpha}$ is not a valid limit of integration. ⓘ

⋯ NIntegrate: $\tau = \sqrt{\alpha}$ is not a valid limit of integration. ⓘ

⋯ General: Further output of NIntegrate::nlim will be suppressed during this calculation. ⓘ

In[508]:=

```
figbLtobii = Plot[
   Table[BDirLiu[d, λα[α, 1], 1] / BDir[d, α, 1], {d, {2, 3, 4, 5}}] // Evaluate,
   {α, 0.7, 1}, PlotRange → {0.978, 1}, Ticks → {xtb, ytb} ];
(* ignore warnings! *)
```

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⋯ General: Further output of NIntegrate::nlim will be suppressed during this calculation. ⓘ

In[509]:=

```
figbLiutob = Show[figbLtobi,
  Graphics[{Arrowheads[0.05], Arrow[{{0.85, 0.99}, {0.6, 0.77}}]}], Epilog →
   Inset[Framed[Show[figbLtobii, ImageSize → 150], ContentPadding → False,
    FrameMargins → Tiny, RoundingRadius → 10], {0.6, 0.6} ]
 ]
```

Out[509]=


```
Export["figbLiutob.pdf", figbLiutob]
```
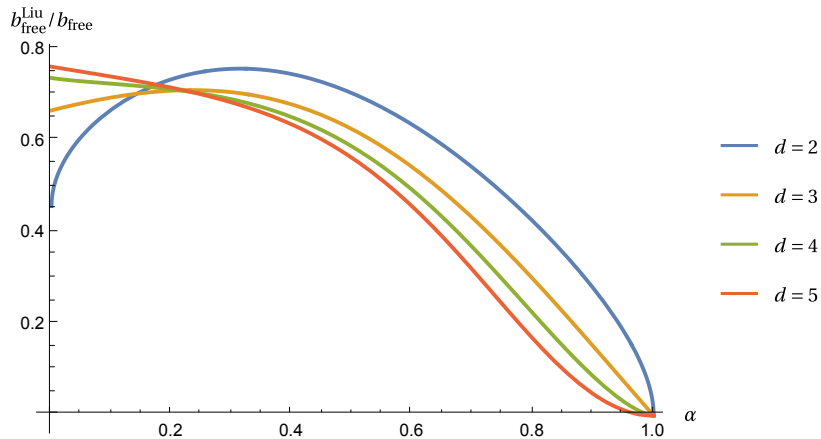
Out[ ]=

figbLiutob.pdf

In[510]:=
```
figbLtobiFree =
 Plot[{-BDirLiu[2, λα[α, 1], 1] / Bfree[2, α, 1], -BDirLiu[3, λα[α, 1], 1] /
     Bfree[3, α, 1], -BDirLiu[4, λα[α, 1], 1] / Bfree[4, α, 1],
   -BDirLiu[5, λα[α, 1], 1] / Bfree[5, α, 1]}, {α, 0, 1}, AxesLabel →
    MaTeX[{"\\alpha", "b^\\mathrm{Liu}_\\mathrm{free}/b_\\mathrm{free}"}],
   PlotLegends → LineLegend[97, MaTeX[{"d=2", "d=3", "d=4", "d=5"}],
    LegendFunction → None, LegendLayout → {"Column", 1}]]
```

Out[510]=

In[ ]:=
```
Export["figbLtobfree.pdf", figbLtobiFree]
```

Out[ ]=

figbLtobfree.pdf

In[512]:=
```
yt2 = Table[{y, MaTeX[y]}, {y, -3, -2.25, 0.25}];
```

In[513]:=
```
figbsi =
  Plot[Table[2 ^ (d + 1) Pi ^ ((d - 1) / 2) BDir[d, α, 1], {d, {2, 3, 4, 5}}] // Evaluate,
    {α, 0, 1}, Ticks → {xt, yt2}, AxesLabel → MaTeX[
     {"\\alpha", "2^{d+1}\\pi^{(d-1)/2} \\mu^{(d-1)/2} b_\\mathrm{Dir}"}] ];
(* ignore warnings! *)
```

⋯ **NIntegrate**: $\tau = \sqrt{\alpha}$ is not a valid limit of integration. ⓘ

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⋯ **General**: Further output of NIntegrate::nlim will be suppressed during this calculation. ⓘ

In[514]:=
```
figgamma = Plot[γR[α], {α, 0, 1}, PlotStyle → Black,
   AxesLabel → MaTeX[{"\\alpha", "\\gamma_R"}], Ticks →
    {Table[{x, MaTeX[x]}, {x, {0.5, 1}}], Table[{x, MaTeX[x]}, {x, {0.5, 1}}]}];
```

In[515]:=
```
yt3 = Table[{y, MaTeX[y]}, {y, 3, 12, 3}];
```
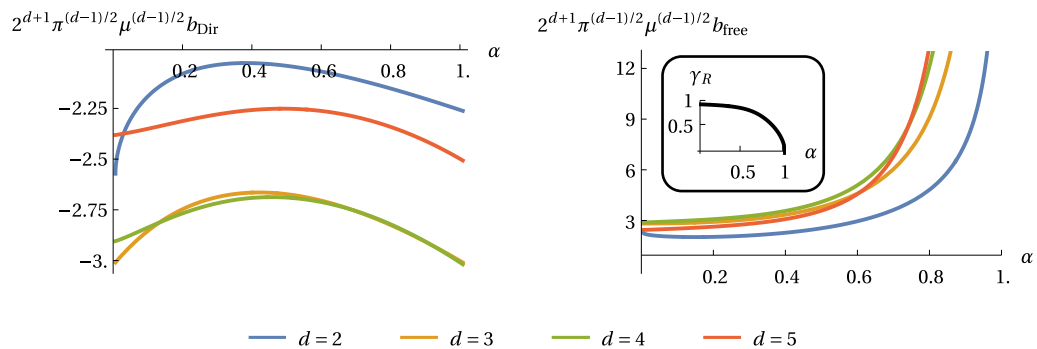
In[518]:=
```
figbsii = Plot[
   {2 ^ (2 + 1) Pi ^ ((2 - 1) / 2) Bfree[2, α, 1],
    2 ^ (3 + 1) Pi ^ ((3 - 1) / 2) Bfree[3, α, 1], 2 ^ (4 + 1) Pi ^ ((4 - 1) / 2) Bfree[4, α, 1],
    2 ^ (5 + 1) Pi ^ ((5 - 1) / 2) Bfree[5, α, 1]}, {α, 0, 1}, Ticks → {xt, yt3},
   AxesOrigin → {0, 1}, PlotRange → {{0, 1}, {0, 13}}, AxesLabel → MaTeX[
     {"\\alpha", "2^{d+1}\\pi^{(d-1)/2} \\mu^{(d-1)/2} b_\\mathrm{free}"}],
   Epilog → Inset[Framed[Show[figgamma, ImageSize → 80],
     ContentPadding → False, FrameMargins → Tiny, RoundingRadius → 10],
     Scaled[{0.05, 1}], Scaled[{0, 1}] ] ];
```

In[517]:=
```
figbs = GraphicsColumn[{GraphicsRow[{figbsi, figbsii}],
   LineLegend[97, MaTeX[{"d=2\\qquad", "d=3\\qquad", "d=4\\qquad", "d=5"}],
   LegendFunction → None, LegendLayout → {"Row", 1}] }]
```

Out[517]=

In[•]:= `Export["figbs.pdf", figbs]`

Out[•]=
figbs.pdf

In[•]:= `γR[0] // N`
`γR[1]`

Out[•]=
0.955313

Out[•]=
0