Dirichlet Laplacian, *Dirichlet–Laplace operator* – In a broad sense, a restriction of the **Laplace operator** to the space of functions satisfying (in some sense) homogeneous **Dirichlet boundary conditions**. For an open set Ω in $\mathbf{R}^n$, the Dirichlet Laplacian is usually defined via the *Friedrichs extension* procedure. Namely, first consider the (negative) Laplace operator $-\Delta$ defined on the subspace $C_0^\infty(\Omega) \subset L_2(\Omega)$ of all infinitely smooth functions with compact support in Ω. This is a symmetric operator, and the associated quadratic form (with the same domain $C_0^\infty(\Omega)$) is given by the **Dirichlet integral**

$$E(f) = \int_\Omega |\nabla f|^2 \, dx \,. \tag{1}$$

Then the form E is closable with respect to the norm

$$\left(E(f) + \|f\|_{L_2(\Omega)}\right)^{1/2} .$$

The domain of its closure $\tilde{E}$ is the **Sobolev space** $H_0^1(\Omega) = W_0^{1,2}(\Omega)$. Then $\tilde{E}$ (given again by the right-hand side of (1)) is the quadratic form of a non-negative **self-adjoint operator** (denoted by $-\Delta_{\mathrm{Dir}}$); moreover,

$$\mathrm{Dom}\left((-\Delta_{\mathrm{Dir}})^{1/2}\right) = \mathrm{Dom}(\tilde{E}) = H_0^1(\Omega) \,.$$

The operator Δ_{Dir} (sometimes taken with the minus sign) is called the *Dirichlet Laplacian* (in the weak sense).

If Ω is bounded domain with boundary $\partial\Omega$ of class C^2, then

$$\mathrm{Dom}\left(-\Delta_{\mathrm{Dir}}\right) = H_0^1(\Omega) \cap H^2(\Omega) \,.$$

The Dirichlet Laplacian for a compact **Riemannian manifold** with boundary is defined similarly.

For a bounded open set Ω in $\mathbf{R}^n$, $-\Delta_{\mathrm{Dir}}$ is a positive unbounded **linear operator** in $L_2(\Omega)$ with a discrete spectrum (cf. also **Spectrum of an operator**). Its eigenvalues $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \le \ldots$ (written in increasing order with account of multiplicity) can be found using the *Rayleigh–Ritz variational formula* (or *max–min formula*)

$$\lambda_n(\Omega) = \inf\{\lambda(L) : L \subseteq C_0^\infty(\Omega), \dim(L) = n\} \,,$$

where

$$\lambda(L) = \sup\{E(f) : f \in L, \|f\|_{L_2(\Omega)} = 1$$

for a finite-dimensional linear subspace L of $C_0^\infty(\Omega)$. It follows from the Rayleigh–Ritz formula that the eigenvalues λ_n are monotonically decreasing functions of Ω. See also [3] for the survey of the asymptotic behaviour of the eigenvalues of the Dirichlet Laplacian and operators corresponding to other boundary value problems for elliptic differential operators.

References

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