APPENDIX B

FOURIER TAUBERIAN THEOREMS

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The objective of this appendix is to formulate and prove Fourier Tauberian theorems as theorems of **classical analysis** without any reference to partial differential equations, spectral theory etc.

The notion of a *Tauberian theorem* covers a wide range of different mathematical results, see the history of the subject in [Ga]. These results have the following in common. Suppose that we have some mathematical object with highly irregular behaviour (say, a discontinuous function or a divergent series) and suppose that we apply some averaging procedure which makes our object substantially more regular (say, a transformation which turns our discontinuous function into an infinitely smooth one or makes our divergent series absolutely convergent). A Tauberian theorem in our understanding is a mathematical result which recovers properties of the original irregular object from the properties of the averaged object.

Tauberian theorems described in this appendix are associated mainly with the Fourier transform. We give four main results — Theorems B.2.1, B.3.1, B.4.1 and B.5.1. Theorem B.2.1 essentially repeats the original Fourier Tauberian Theorem of B.M. Levitan, see also [Hö3, vol. 3, Lemma 17.5.6]. Theorem B.3.1 provides a useful rough estimate. Theorem B.4.1 is a refined (with improved remainder estimate) theorem of the type introduced by Safarov [Sa2]–[Sa6] for studying general (i.e., not necessarily polynomial) spectral asymptotics. Finally, Theorem B.5.1 is a special version of Theorem B.4.1 specifically designed for studying polynomial two-term spectral asymptotics; a theorem of this sort was implicitly used by Duistermaat and Guillemin in their pioneering paper [DuiGui].

Hereinafter in this appendix we use the "hat" to denote the Fourier transform of a function. By a prime we denote the derivative.

B.1. Introductory remarks

It is well known that, under certain conditions, the asymptotic behaviour of a function at infinity is determined by the singularities of its Fourier transform. We illustrate this fact by the following elementary example.

Example B.1.1. Let $\hat f(t)$ be a complex-valued function on $\mathbb{R}$ which is infinitely smooth outside the point $t=0$, at which the function $\hat f(t)$ together with all its derivatives has finite left and right limits. We denote

$$\delta_k = \hat f^{(k)}(+0) - \hat f^{(k)}(-0), \qquad k = 0,1,\dots$$

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If $\hat{f}(t)$ and all its derivatives vanish faster than any given negative power of $|t|$ as $t \to \infty$ then the inverse Fourier transform $f(\lambda) = \mathcal{F}^{-1}_{t\to\lambda}[\hat{f}(t)]$ admits the following asymptotic expansion:

$$f(\lambda) \sim \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{\delta_k}{(-i\lambda)^{k+1}}\,, \qquad \lambda \to \infty\,. \tag{B.1.1}$$

We shall consider a more complicated situation. First, we allow $\hat{f}$ to be a distribution. Second, we allow $\hat{f}$ to have singularities not only at the origin. Third, we assume that information on $\hat{f}(t)$ is given only on a finite time interval rather than for all $t \in \mathbb{R}$. Under these assumptions it is impossible to construct a full asymptotic expansion of the type (B.1.1). Nevertheless, it turns out that for a monotone function f one can still relate the behaviour of $f(\lambda)$ at infinity to the singularities of its Fourier transform. This relation is the subject of the Fourier Tauberian theorems formulated in this appendix.

We shall denote by F_+ the class of real-valued monotone non-decreasing functions N on $\mathbb{R}$ such that $N(\lambda) = 0$ for $l \leqslant 0$, and $\lambda^{-p}N(\lambda) \to 0$ as $\lambda \to +\infty$, where p is some positive number depending on the particular function N. The latter condition can be rewritten as

$$N(\lambda) = o(\lambda^p)\,, \qquad \lambda \to +\infty\,. \tag{B.1.2}$$

Let us fix a real-valued function ρ on $\mathbb{R}$ satisfying the following five conditions:

(1) $\rho \in \mathcal{S}(\mathbb{R})$;
(2) $\rho(\lambda) > 0$ for all $\lambda \in \mathbb{R}$;
(3) $\hat{\rho}(0) = \int \rho(\lambda)\, d\lambda = 1$;
(4) $\operatorname{supp}\hat{\rho}$ is compact;
(5) the function $\rho(\lambda)$ is even.

Such a function ρ exists (see, for instance, [Hö3, vol. 3, Sect. 17.5]). Note that under conditions (1)–(5) the Fourier transform $\hat{\rho}$ is a real even function.

We shall denote $\rho_T(\lambda) := T\rho(T\lambda)$, $\hat{\rho}_T(t) := \hat{\rho}(t/T)$, where T is a positive parameter. Obviously, if the function ρ satisfies the five conditions stated above then ρ_T satisfies these conditions as well.

Throughout this appendix ν will denote some real number, not necessarily positive. All subsequent asymptotics in this appendix are written with respect to $\lambda \to +\infty$. Of course, asymptotics with respect to $\lambda \to -\infty$ are of no interest because for all $N \in F_+$ we have $N(\lambda) \equiv 0$ and $(N * \rho)(\lambda) = O(|\lambda|^{-\infty})$ as $\lambda \to -\infty$.

B.2. Basic theorem

In this section we will be using the notation C for various positive constants which may depend only on the choice of the function ρ and on the number ν but are independent of the function N.

THEOREM B.2.1. *If $N \in F_+$, and*

$$(N' * \rho)(\lambda) \leqslant \lambda^\nu\,, \qquad \forall \lambda \geqslant 1\,, \tag{B.2.1}$$

then

$$|N(\lambda) - (N * \rho)(\lambda)| \leqslant C\,\lambda^\nu\,, \qquad \forall \lambda \geqslant 1\,. \tag{B.2.2}$$

By an elementary rescaling argument Theorem B.2.1 immediately implies

Corollary B.2.2. *If* $N \in F_+$ *and* $(N' * \rho)(\lambda) = O(\lambda^\nu)$ *then*

$$N(\lambda) \;=\; (N * \rho)(\lambda) \;+\; O(\lambda^\nu)\,. \tag{B.2.3}$$

Moreover, if the estimate $(N' * \rho)(\lambda) = O(\lambda^\nu)$ *holds uniformly on some subset of* F_+*, then (B.2.3) is also uniform.*

Clearly, formula (B.2.3) can be rewritten as

$$N(\lambda) \;=\; \mathcal{F}^{-1}_{t\to\lambda}[\hat\rho(t)\hat N(t)] + O(\lambda^\nu)\,,$$

which means that, as promised, we have established the relation between the behaviour of $N(\lambda)$ at infinity and the behaviour of $\hat N(t)$ in the neighbourhood of the origin.

The proof of Theorem B.2.1 is based on the following technical lemma.

Lemma B.2.3. *Under the conditions of Theorem B.2.1*

$$|N(\lambda+s) - N(\lambda)| \leqslant C(1+|s|)^{1+|\nu|}\lambda^\nu\,, \qquad \forall \lambda \geqslant 1\,, \tag{B.2.4}$$

uniformly over all $s \in \mathbb{R}$*.*

Proof. It is sufficient to prove (B.2.4) in the case when λ and s are integers. Indeed, the general case is reduced to this one by perturbing λ and s to one of their two nearest integers and using the monotonicity of N. Note also that the case $s=0$ is trivial. Thus, in order to prove the lemma it is sufficient to establish the following two estimates:

$$N(\lambda+s) - N(\lambda) \leqslant Cs^{1+|\nu|}\lambda^\nu\,, \qquad \forall \lambda, s \in \mathbb{N}\,, \tag{B.2.5}$$

$$N(\lambda) - N(\lambda-s) \leqslant Cs^{1+|\nu|}\lambda^\nu\,, \qquad \forall \lambda, s \in \mathbb{N}\,. \tag{B.2.6}$$

We have

$$\begin{aligned}(N' * \rho)(\lambda) \;=\; \int \rho(\lambda-\mu)\,dN(\mu) \;\geqslant\; C\int_{\lambda-1}^{\lambda+1} dN(\mu)\\ \;=\; C\big((N(\lambda+1)-N(\lambda)) + (N(\lambda)-N(\lambda-1)\big)\end{aligned} \tag{B.2.7}$$

(here we have used the fact that $\rho(\Lambda) \geqslant C > 0$ for $\Lambda \in [-1,1]$). Formulae (B.2.1), (B.2.7) imply

$$N(\lambda+1) - N(\lambda) \leqslant C\,\lambda^\nu\,, \qquad \forall \lambda \in \mathbb{N}\,, \tag{B.2.8}$$

$$N(\lambda) - N(\lambda-1) \leqslant C\,\lambda^\nu\,, \qquad \forall \lambda \in \mathbb{N}\,. \tag{B.2.9}$$

Adding up the inequalities (B.2.8) we get

$$N(\lambda+s)-N(\lambda)=\sum_{k=1}^{s}(N(\lambda+k)-N(\lambda+k-1))$$
$$\leqslant C\sum_{k=1}^{s}(\lambda+k-1)^\nu \leqslant \begin{cases} Cs(\lambda+s-1)^\nu\,, & \text{if} \quad \nu\geqslant 0\,,\\ Cs\lambda^\nu\,, & \text{if} \quad \nu<0\,.\end{cases}$$

Since in the case $\nu \geqslant 0$ we have $(\lambda+s-1)^\nu \leqslant s^\nu\lambda^\nu$, the above formula implies (B.2.5).

Let now prove (B.2.6). It is sufficient to prove (B.2.6) for $s\leqslant\lambda$ because for $s>\lambda$ we have $N(\lambda)-N(\lambda-s)=N(\lambda)-N(\lambda-\lambda)$. So further on $s\leqslant\lambda$. Adding up the inequalities (B.2.9) we get

$$N(\lambda)-N(\lambda-s)=\sum_{k=1}^{s}(N(\lambda-k+1)-N(\lambda-k))$$
$$\leqslant C\sum_{k=1}^{s}(\lambda-k+1)^\nu \leqslant \begin{cases} Cs\lambda^\nu\,, & \text{if} \quad \nu\geqslant 0\,,\\ Cs(\lambda-s+1)^\nu\,, & \text{if} \quad \nu<0\,.\end{cases}$$

Since in the case $\nu<0$ we have $(\lambda-s+1)^\nu\leqslant s^{|\nu|}\lambda^\nu$, the above formula implies (B.2.6). $\square$

Proof of Theorem B.2.1. Using (B.2.4) we obtain

$$|(N*\rho)(\lambda)-N(\lambda)| \,=\, \left|\int (N(\lambda-\mu)-N(\lambda))\,\rho(\mu)\,d\mu\right|$$
$$\leqslant\,\int |N(\lambda-\mu)-N(\lambda)|\,\rho(\mu)\,d\mu \,\leqslant\, C\,\lambda^\nu\int(1+|\mu|)^{1+|\nu|}\,\rho(\mu)\,d\mu \,=\, C\,\lambda^\nu$$

for all $\lambda\geqslant 1$. $\square$

The following is a weighted version of Theorem B.2.1.

Theorem B.2.4. *Let* $N\in F_+$ *and*

$$(N'*\rho_T)(\lambda)\,\leqslant\,d\,\lambda^\nu\,,\qquad \forall\lambda\geqslant T^{-1}\,, \tag{B.2.10}$$

where $d>0$, $T>0$ *are parameters. Then*

$$|N(\lambda)\,-\,(N*\rho_T)(\lambda)|\,\leqslant\,C\,d\,T^{-1}\lambda^\nu\,,\qquad \forall\lambda\geqslant T^{-1}\,. \tag{B.2.11}$$

Here the constant C *is the same as in Theorem B.2.1.*

Proof. Set $\widetilde\lambda:=T\lambda$, $\widetilde N(\widetilde\lambda):=d^{-1}T^{1+\nu}N(\widetilde\lambda/T)$. Then (B.2.10) is equivalent to

$$(\widetilde N'*\rho)(\widetilde\lambda)\,\leqslant\,\widetilde\lambda^\nu\,,\qquad\forall\widetilde\lambda\geqslant 1\,.$$

Therefore by Theorem B.2.1

$$|\widetilde N(\widetilde\lambda)\,-\,(\widetilde N*\rho)(\widetilde\lambda)|\,\leqslant\,C\,\widetilde\lambda^\nu\,,\qquad\forall\widetilde\lambda\geqslant 1\,.$$

The latter is equivalent to (B.2.11). $\square$

B.3. Rough estimate for the nonzero singularities

In this section we give a simple result which shows that the singularities of the distribution $\mathcal{F}_{\lambda\to t}[N'(\lambda)]$ at different t are not totally independent. Namely, we show that the singularities at $t \neq 0$ cannot be stronger than the singularity at $t = 0$.

THEOREM B.3.1. *If* $N \in F_+$ *and*

$$(N' * \rho)(\lambda) = O(\lambda^\nu), \tag{B.3.1}$$

then for any function γ *such that* $\hat{\gamma} \in C_0^\infty(\mathbb{R})$ *we have*

$$(N' * \gamma)(\lambda) = O(\lambda^\nu), \tag{B.3.2}$$

and moreover

$$\limsup_{\lambda\to+\infty} \frac{|(N' * \gamma)(\lambda)|}{\lambda^\nu} \leqslant C_{\rho,\gamma} \limsup_{\lambda\to+\infty} \frac{(N' * \rho)(\lambda)}{\lambda^\nu}, \tag{B.3.3}$$

where $C_{\rho,\gamma} > 0$ *is a constant independent of the choice of the function* N *and of the number* ν.

PROOF. Let us choose a number $\delta > 0$ such that $\hat{\rho}(t) \neq 0$ on $[-\delta, \delta]$. Further on we assume without loss of generality that $\operatorname{diam}(\operatorname{supp}\hat{\gamma}) \leqslant 2\delta$. Indeed, this can always be achieved by representing the original function γ as a finite sum of functions possessing this property.

Let us choose a $\tau \in \mathbb{R}$ such that $\operatorname{supp}\hat{\gamma} \in [\tau - \delta, \tau + \delta]$ and denote

$$\alpha(\lambda) = e^{i\tau\lambda}\rho(\lambda), \quad \beta(\lambda) = \mathcal{F}^{-1}_{t\to\lambda}\left[\frac{\hat{\gamma}(t)}{\hat{\alpha}(t)}\right].$$

Then

$$(N' * \gamma)(\lambda) = ((N' * \alpha) * \beta)(\lambda). \tag{B.3.4}$$

As ρ is a nonnegative function and dN is a nonnegative measure we have

$$|(N' * \alpha)(\lambda)| \leqslant (N' * \rho)(\lambda). \tag{B.3.5}$$

Formulae (B.3.4), (B.3.5), (B.3.1) imply (B.3.3) with $C_{\rho,\gamma} = \int |\beta(\mu)|\, d\mu$. $\square$

B.4. General refined theorem

In this section we denote by C positive constants (maybe, different) which may depend only on the functions N_j, ρ and on the number ν, but not on the parameters s, T and ε. If a constant depends on some of the parameters s, T or ε this will be indicated by respective subscripts.

THEOREM B.4.1. *Let* $N_j \in F_+$, $(N_j' * \rho)(\lambda) = O(\lambda^\nu)$, $j = 1, 2$,

$$(N_2 * \rho)(\lambda) = (N_1 * \rho)(\lambda) + o(\lambda^\nu) \tag{B.4.1}$$

and

$$(N_2' * \gamma)(\lambda) = (N_1' * \gamma)(\lambda) + o(\lambda^\nu) \tag{B.4.2}$$

for any function γ *such that* $\hat\gamma \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\hat\gamma \subset (0, +\infty)$. *Then*

$$N_1(\lambda - \varepsilon) - o(\lambda^\nu) \leqslant N_2(\lambda) \leqslant N_1(\lambda + \varepsilon) + o(\lambda^\nu), \qquad \forall \varepsilon > 0,$$

or equivalently

$$N_1(\lambda - o(1)) - o(\lambda^\nu) \leqslant N_2(\lambda) \leqslant N_1(\lambda + o(1)) + o(\lambda^\nu). \tag{B.4.3}$$

Formula (B.4.3) means that there exists a (positive) function f such that $f(\lambda) \to 0$ as $\lambda \to +\infty$, and

$$N_1(\lambda - f(\lambda)) - \lambda^\nu f(\lambda) \leqslant N_2(\lambda) \leqslant N_1(\lambda + f(\lambda)) + \lambda^\nu f(\lambda).$$

Formula (B.4.3) differs from standard asymptotic formulae (like (B.2.3) above or (B.5.2) below) in the sense that we are comparing the graphs of the functions $N_1(\lambda)$ and $N_2(\lambda)$ not only in the "vertical" direction, but in the "horizontal" direction as well. This is natural when one compares the graphs of two counting functions: trying to increase accuracy we inevitably have to start comparing the graphs in both directions due to the possible presence of discontinuities. This phenomenon is well known in probability theory, where graphs of monotone functions are compared using the so-called *Lévy metric*, [GnKo, Chapter 2, Section 9]. The basic idea in the definition of the classical Lévy metric is to measure the distance between the graphs in the direction forming an angle of $\frac{3\pi}{4}$ with the positive λ-semiaxis. Theorem B.4.1 can be reformulated in terms of a weighted Lévy metric, in which the angle is a function of λ.

We shall prove Theorem B.4.1 in several steps. First, we prove

LEMMA B.4.2. *Let* $N \in F_+$ *and* $(N' * \rho)(\lambda) = O(\lambda^\nu)$. *Then for all* $s \geqslant 0$, $T \geqslant 1$, $\lambda \geqslant 1$ *we have the uniform estimate*

$$(N*\rho_T)(\lambda-s) - C(1+sT)^{-1}\lambda^\nu \leqslant N(\lambda) \leqslant (N*\rho_T)(\lambda+s) + C(1+sT)^{-1}\lambda^\nu. \tag{B.4.4}$$

PROOF. For the sake of brevity, we shall prove only the left inequality (B.4.4); the right one is proved in a similar way.

We have

$$\begin{aligned}(N * \rho_T)(\lambda - s) - N(\lambda) &= \int \big(N(\lambda - s - T^{-1}\tau) - N(\lambda)\big)\,\rho(\tau)\,d\tau \\ &\leqslant \int_{-\infty}^{-sT} \big(N(\lambda - s - T^{-1}\tau) - N(\lambda)\big)\,\rho(\tau)\,d\tau\end{aligned}$$

(here we used the monotonicity of N). Estimating the integrand by Lemma B.2.3 we obtain

$$(N * \rho_T)(\lambda - s) - N(\lambda) \leqslant C\,\lambda^\nu \int\limits_{sT}^{+\infty} (1 - s + T^{-1}\tau)^{1+|\nu|} \rho(\tau)\, d\tau \tag{B.4.5}$$

for $\lambda \geqslant 1$. But

$$(1 - s + T^{-1}\tau)^{1+|\nu|} \leqslant (1+\tau)^{1+|\nu|}, \qquad \forall \tau \geqslant sT$$

(here we used the inequalities $s \geqslant 0$, $T \geqslant 1$), and

$$\rho(\tau) \leqslant C\,(1+\tau)^{-3-|\nu|}, \qquad \forall \tau \geqslant 0,$$

so

$$\int\limits_{sT}^{+\infty} (1 - s + T^{-1}\tau)^{1+|\nu|} \rho(\tau)\, d\tau = C \int\limits_{sT}^{+\infty} (1+\tau)^{-2}\, d\tau = C(1+sT)^{-1}.$$

Substituting the latter into (B.4.5) we arrive at (B.4.4). $\square$

REMARK B.4.3. Obviously, if the estimate $(N' * \rho)(\lambda) = O(\lambda^\nu)$ holds uniformly on some subset of F_+, then the constant C in (B.4.4) is independent of the particular function N from this subset (i.e., (B.4.4) is also uniform).

Lemma B.4.2 implies

LEMMA B.4.4. *Let* $N_j \in F_+$, $(N_j' * \rho)(\lambda) = O(\lambda^\nu)$, $j = 1, 2$, *and*

$$(N_2 * \rho_T)(\lambda) = (N_1 * \rho_T)(\lambda) + o(\lambda^\nu), \qquad \forall T \geqslant 1. \tag{B.4.6}$$

Then (B.4.3) holds.

PROOF. Note that both functions N_2 and N_1 satisfy the conditions of Lemma B.4.2, and therefore estimates (B.4.4) hold. With account of (B.4.6) the inequalities (B.4.4) can be rewritten as

$$\begin{aligned} (N_k * \rho_T)(\lambda - s) - C(1+sT)^{-1}\lambda^\nu - o((\lambda - s)^\nu) \\ \leqslant N_j(\lambda) \leqslant \\ (N_k * \rho_T)(\lambda + s) + C(1+sT)^{-1}\lambda^\nu + o((\lambda + s)^\nu), \end{aligned} \tag{B.4.7}$$

where $j, k = 1, 2$, $j \neq k$. Formulae (B.4.7) imply

$$\begin{aligned} N_1(\lambda - 2s) - C(1+sT)^{-1}(\lambda^\nu + (\lambda - s)^\nu) - o((\lambda - s)^\nu) \\ \leqslant N_2(\lambda) \leqslant \\ N_1(\lambda + 2s) + C(1+sT)^{-1}(\lambda^\nu + (\lambda + s)^\nu) + o((\lambda + s)^\nu). \end{aligned} \tag{B.4.8}$$

Formula (B.4.8) holds for any fixed $s \geqslant 0$ and $T \geqslant 1$. So we can set $s = T^{-1/2}$. Then (B.4.8) takes the form

$$
\begin{aligned}
\text{(B.4.9)}\quad N_1(\lambda - 2T^{-1/2}) - CT^{-1/2}\left(\lambda^\nu + (\lambda - T^{-1/2})^\nu\right) - o\left((\lambda - T^{-1/2})^\nu\right) \\
\leqslant N_2(\lambda) \leqslant \\
N_1(\lambda + 2T^{-1/2}) + CT^{-1/2}\left(\lambda^\nu + (\lambda + T^{-1/2})^\nu\right) + o\left((\lambda + T^{-1/2})^\nu\right).
\end{aligned}
$$

It remains only to set $T = T(\lambda)$, where $T(\lambda)$ is an increasing function which tends to $+\infty$ as $\lambda \to +\infty$. The function $T = T(\lambda)$ can be chosen to increase so slowly that for both o-terms appearing in (B.4.9) we have $o\left((\lambda \mp T^{-1/2}(\lambda))^\nu\right) = o(\lambda^\nu)$; this remark is necessary because our o-terms (originating from (B.4.6)) might depend on T.

Formula (B.4.9) with the substitution $T = T(\lambda)$ implies (B.4.3). $\square$

Proof of Theorem B.4.1. In view of Lemma B.4.4, it is sufficient to show that (B.4.1) and (B.4.2) imply (B.4.6). Let us split $\hat\rho_T$ into the sum of functions $\hat\rho_{T,1}, \hat\rho_{T,2} \in C_0^\infty(\mathbb{R})$ such that $\hat\rho(t) \geqslant C > 0$ for $t \in \operatorname{supp}\hat\rho_{T,1}$, and $\hat\rho_{T,2}(t)$ vanishes in a neighbourhood of $t = 0$. Then (B.4.2) implies (B.4.6) for $\rho_{T,2}$ because

$$(N_j * \rho_{T,2})(\lambda) \;=\; \mathcal{F}^{-1}_{t\to\lambda}\left[(it)\hat N_j(t)\,(it)^{-1}\hat\rho_{T,2}(t)\right] \;=\; (N_j' * \widetilde\rho_{T,2})(\lambda)$$

where $j = 1, 2$, and $\widetilde\rho_{T,2}(\lambda) = \mathcal{F}^{-1}_{t\to\lambda}\left[(it)^{-1}\hat\rho_{T,2}(t)\right]$. Obviously,

$$\text{(B.4.10)}\qquad N_j * \rho_{T,1} \;=\; N_j * \rho * \beta\,,\qquad \beta(\lambda) = \mathcal{F}^{-1}_{t\to\lambda}\left[\frac{\hat\rho_{T,1}(t)}{\hat\rho(t)}\right],\quad j = 1, 2\,.$$

Since the function β is rapidly decreasing, (B.4.1) and (B.4.10) imply (B.4.6) for $\rho_{T,1}$. $\square$

B.5. Special version of the general refined theorem

We formulate below a special version of the general refined theorem specifically oriented towards the case when N has a polynomial two-term asymptotics, i.e., when the nonzero singularities of the Fourier transform of N' are weaker than the singularity at the origin.

Theorem B.5.1. *Let $N \in F_+$, $(N' * \rho)(\lambda) = O(\lambda^\nu)$, and*

$$\text{(B.5.1)}\qquad (N' * \gamma)(\lambda) \;=\; o(\lambda^\nu)\,,$$

for any function γ such that $\hat\gamma \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\hat\gamma \subset (0, +\infty)$. Then

$$\text{(B.5.2)}\qquad N(\lambda) \;=\; (N * \rho)(\lambda) \;+\; o(\lambda^\nu)\,.$$

Remark B.5.2. As will be clear from the proof, if the conditions of Theorem B.5.1 are fulfilled uniformly on some subset of F_+ then (B.5.2) also holds uniformly.

Theorem B.5.1 has a simpler formulation than Theorem B.4.1, and it can be viewed as a natural extension of Corollary B.2.2. However, Theorem B.5.1 requires more restrictive conditions on the functions involved than Theorem B.4.1. Basically, the conditions of Theorem B.5.1 ensure that the (discontinuous) monotone function N does not have very big "jumps", and this is why we are able to compare the graphs in the "vertical" direction only.

Proof of Theorem B.5.1. According to the Lagrange formula we have for some $\theta_\mp(\lambda) \in (0,1)$

(B.5.3) $$(N*\rho)(\lambda \mp o(1)) - (N*\rho)(\lambda) = o(1)\,(N'*\rho)\,(\lambda \mp \theta_\mp(\lambda)\,o(1)) = o(1)\,O(\lambda^\nu) = o(\lambda^\nu)\,.$$

In view of Lemma B.4.2 and (B.5.3), it is sufficient to prove that

(B.5.4) $$(N*\rho)(\lambda) - (N*\rho_T)(\lambda) = (N*(\rho-\rho_T))\,(\lambda) = o(\lambda^\nu)\,, \qquad \forall T \geqslant 1\,.$$

Indeed, then (B.5.2) is obtained from (B.4.4) by substitution $s = f_1(\lambda)$, $T = f_2(\lambda)$, where f_1 and f_2 are arbitrary functions such that

$$f_1(\lambda) \to 0\,, \quad f_2(\lambda) \to +\infty\,, \quad f_1(\lambda)\,f_2(\lambda) \to +\infty\,, \qquad \lambda \to +\infty\,.$$

Since $\hat\rho(0) = 1$ and $\hat\rho'(0) = 0$, the function $\hat\rho(t) - \hat\rho_T(t)$ has a second order zero at $t = 0$. Therefore (B.5.4) is a consequence of the following lemma, which completes the proof of Theorem B.5.1.

Lemma B.5.3. *Let $N \in F_+$ satisfy the conditions of Theorem B.5.1. Assume that $\hat\rho_0 \in C_0^\infty(\mathbb{R})$ and $\hat\rho_0(t) = t^2\hat\alpha(t)$, where $\hat\alpha \in C_0^\infty(\mathbb{R})$. Then*

(B.5.5) $$(N*\rho_0)(\lambda) = o(\lambda^\nu)\,.$$

Proof. We have

(B.5.6) $$(N*\rho_0)(\lambda) = \mathcal{F}^{-1}_{t\to\lambda}[\hat N(t)\,t^2\hat\alpha(t)] = (N'*\beta)(\lambda)\,,$$

where $\beta = \mathcal{F}^{-1}_{t\to\lambda}[-it\,\hat\alpha(t)]$.

Let $\hat f \in C_0^\infty(\mathbb{R})$ and $\hat f(t) = 1$ in a neighbourhood of origin. Denote $\hat f_\delta(t) = \hat f(t/\delta)$ and $\widetilde{f_\delta}(\lambda) = \mathcal{F}^{-1}_{t\to\lambda}[-it\,\hat f_\delta(t)]$. Then $\widetilde{f_\delta}(\lambda) = \delta^2\,\mathcal{F}^{-1}_{t\to\delta\lambda}[-it\,\hat f(t)]$ and

$$\int |\widetilde{f_\delta}(\lambda)|\,d\lambda = \delta\,C_f\,,$$

where $C_f = \int |\mathcal{F}^{-1}_{t\to\lambda}[-it\,\hat f(t)]\,|\,d\lambda$.

Clearly,

(B.5.8) $$\beta(\lambda) = \mathcal{F}^{-1}_{t\to\lambda}[-it\,(1-\hat f_\delta(t))\,\hat\alpha(t)] + \mathcal{F}^{-1}_{t\to\lambda}[-it\,\hat f_\delta(t)\,\hat\alpha(t)]\,.$$

In view of (B.5.1), the contribution to (B.5.6) of first term in the right-hand side of (B.5.8) is $o(\lambda^n)$. The contribution to (B.5.6) of the second term in the right-hand side of (B.5.8) is $(N'*\alpha*f_\delta)$. By Theorem B.3.1 $(N'*\alpha)(\lambda)$ is estimated by $C\,(\lambda^\nu+1)$, so

$$|\,(N'*\alpha*f_\delta)(\lambda)\,| \leqslant \delta\,C\,C_f\,(\lambda^\nu+1)\,,$$

Thus,

$$|\,(N*\rho_0)(\lambda)\,| \leqslant \delta\,C\,C_f\,(\lambda^\nu+1) + o(\lambda^n)\,.$$

Since δ can be chosen arbitrarily small, this implies (B.5.5). $\square$