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Spectral Problems for the Lamé System with Spectral Parameter in Boundary Conditions on Smooth or Nonsmooth Boundary*

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Abstract. The paper is devoted to four spectral problems for the Lamé system of linear elasticity in domains of $\mathbb{R}^3$ with compact connected boundary S. The frequency is fixed in the upper closed half-plane; the spectral parameter enters into the boundary or transmission conditions on S. Two cases are investigated: (1) S is C^∞; (2) S is Lipschitz.

INTRODUCTION

In this paper we consider four spectral problems for the Lamé system of linear elasticity, see (1.3). The system contains the frequency parameter ω, which is a fixed complex number with $\operatorname{Re}\omega \geqslant 0$. The statements of Problems I–IV are given in Subsection 1.1. The spectral parameter ζ enters into the boundary conditions (in Problems I, II) or transmission conditions (in Problems III, IV) on a closed connected surface S, which divides its complement into a bounded domain G^+ and an unbounded domain G^-. This surface is assumed to be infinitely smooth in Section 1 and Lipschitz in Section 2. Our aim is to study the spectral properties of Problems I–IV, including the localization of eigenvalues and their asymptotics, the smoothness of the eigenfunctions or, in the case of nonselfadjoint Problems II–IV, of the root functions, as well as the completeness of these functions and the summability of Fourier series with respect to them.

More precisely, we study the spectral properties of some integral operators on S that arise after the reduction of Problems I–IV to equations on S (see Subsection 1.4) under the assumption that the frequency ω is in some sense nonexceptional. These operators are closely associated with the Lamé system (1.3). The first two of the operators, $T^+(\omega)$ and $T^-(\omega)$, map the boundary data $Tu^\pm$ of the interior or exterior second (or Traction, or Neumann) boundary value problem into the boundary data $u^\pm$ of the first (or Dirichlet) boundary value problem, with the inverse sign in case of $T^+(\omega)$. Here T is the stress operator on S (see (1.5)) in the direction of the exterior unit normal $\nu(x)$. The third operator $A(\omega)$ is the restriction to S of the single layer potential $\mathcal{A}(\omega)$. The fourth operator $T(\omega)$ is the inverse of $-T\mathcal{B}(\omega)$, where $\mathcal{B}(\omega)$ is the double layer potential. See Subsection 1.4 and formula (1.107).

The spectral problems I–IV are similar to four spectral problems for the Helmholtz equation proposed by physicists [34] and analyzed mathematically in [1] in the case of smooth S and in [4] in the case of Lipschitz S (see [4] for more references). The deep analogy between the results for the Helmholtz equation and for the Lamé system suggests that in a similar manner one can consider more general second order partial differential equations and systems of equations; we hope to devote a separate paper to such generalizations. However, the case of the Lamé system is of interest in itself and is technically more difficult than the case of the Helmholtz equation even if the boundary is smooth.

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In the case of a smooth surface S, we use the classical theory of the first and the second boundary value problems of elasticity and the corresponding potential theory described in [22]. However, for our purposes it is more convenient to use the tools of the theory of elliptic pseudodifferential operators (ΨDOs) in Sobolev L_2-spaces $H^t(S)$, $t \geqslant 0$, and in the beginning of Section 1 we reformulate some well-known results in this spirit. In particular, we show that each of the operators $T^{\pm}(\omega)$, $A(\omega)$, and $T(\omega)$ is an elliptic ΨDO of order -1. These operators are either selfadjoint in $L_2(S) = H^0(S)$ (when $\operatorname{Re}\omega = 0$, and, in the case of $T^+(\omega)$, also when $\operatorname{Im}\omega = 0$) or very close to selfadjoint operators, namely, to their real parts (moreover, for the real frequencies ω the operators $T^-(\omega)$, $A(\omega)$, and $T(\omega)$ are "infinitely close" to their real parts). Using these facts, in Subsections 1.8–1.10 we obtain the main theorems of Section 1, which describe the localization and the asymptotics of the characteristic numbers and the fast decrease of their imaginary parts, as well as the infinite smoothness, the completeness of the root functions, and the unconditional convergence of Fourier series with parentheses with respect to these functions. See the precise statements in these subsections.

There is no classical potential theory or calculus of ΨDOs in the case of a Lipschitz S, and the counterparts of some classical facts related to potentials and to the main boundary value problems have been much harder to obtain. In the case $\omega = 0$, this was done in [10], which had been preceded by the study of singular integral operators on Lipschitz surfaces in [6, 7] and the study of the Laplace equation in [32, 33] (see detailed references in [10, 33] and the surveys [20, 21]). First, we extend the results of [10] to the case of nonreal ω, and then, to the case of real nonzero ω. This approach is similar to [31, 25], where the results of [32, 33] for the Laplace equation are extended to the case of the Helmholtz equation. We also use the deep results of [11] for $\omega = 0$. Some additional considerations (e.g., see our Theorems 2.1 and 2.15) have analogs in [4].

Our results concerning the spectral properties of the operators in question in the case of a Lipschitz surface S are of course weaker than in the case of smooth S. Still, we estimate from below the characteristic numbers of our operators and show that the ratios of their imaginary parts to their real parts tend to zero. We also show that the root functions form a complete system in $H^0(S)$ and that the corresponding Fourier series are summable by the Abel–Lidskiĭ method. On a Lipschitz surface, the Sobolev spaces $H^t(S)$ are defined intrinsically only for $|t| \leqslant 1$, and concerning the smoothness of the root functions corresponding to nonzero eigenvalues we show that usually these functions belong to $H^1(S)$.

As in [4], we single out the case of almost smooth surfaces S. A Lipschitz surface S is called *almost smooth* if it is C^∞ outside a closed subset of zero Lebesgue measure called a *singular set.* This definition [3] covers the cases of surfaces with conical points, edges, etc. For an almost smooth surface we use the results of [3] to obtain the asymptotics of the characteristic numbers of our operators (but without an additional remainder estimate); the root functions on such surfaces S are C^∞ outside the singular set. More complete statements are given in Subsections 2.9–2.10, which contain the main results of Section 2.

As we show in Subsections 1.6 and 2.7, our assumption that the frequency ω is not exceptional is equivalent to the condition that $\rho\omega^2$ lies outside the spectra of the first or the second boundary value problems for the Lamé equation. We essentially remove these restrictions in Subsections 1.11 and 2.11.

1. DOMAINS WITH SMOOTH BOUNDARY

1.1. *Statement of the Spectral Problems*

Let S be a closed (i.e., compact and without boundary) connected surface in $\mathbb{R}^3$ which divides $\mathbb{R}^3 \setminus S$ into a bounded domain G^+ and an unbounded domain G^-. In this section S is assumed to be infinitely smooth. Let $L = L(\partial_x)$ be the Lamé operator of classical elasticity. It is the 3×3 matrix partial differential operator with entries

$$L_{jk} = L_{jk}(\partial_x) = \delta_{jk}\mu\Delta + (\lambda + \mu)\partial_j\partial_k \qquad (j, k = 1, 2, 3). \tag{1.1}$$

Here and in what follows $\partial_j = \partial/\partial x_j$, Δ is the scalar Laplacian, and δ_{jk} is Kronecker's delta. The Lamé coefficients λ and μ obey the standard inequalities

$$\mu > 0\,, \qquad 3\lambda + 2\mu > 0. \tag{1.2}$$

We consider the equation of free elastic vibrations

$$L(\partial_x, \omega)u(x) \equiv L(\partial_x)u(x) + \rho\omega^2 u(x) = 0 \tag{1.3}$$

in G^+, G^-, or $G^+ \cup G^-$. In G^- the solution must satisfy the radiation conditions; we recall them below. Here $u(x)$ is the vector of elastic displacements, $\rho = \text{const} > 0$ is the density, and $\omega = \text{const}$ is the vibration frequency. We always assume that $\operatorname{Im}\omega \geqslant 0$:

$$\omega = \omega' + i\omega'', \qquad \omega'' \geqslant 0. \tag{1.4}$$

For $\omega = 0$, we have an equilibrium state, for real $\omega \neq 0$, steady oscillations and for $\omega'' > 0$, $\omega' \neq 0$, damped oscillations. Throughout this paper, when mentioning "all ω," we always mean "all ω from the closed upper complex half-plane."

The Lamé operator is elliptic (see Subsection 1.5 below); therefore, the solutions (in the sense of distributions) of (1.3) in G^+ or G^- are infinitely smooth.

Let $n(x)$ be a unit vector at a point x; we denote by $T = T(\partial_x, n(x))$ the stress operator at x in the direction of $n(x)$. It is the 3×3 matrix partial differential operator with entries

$$T_{jk} = T_{jk}(\partial_x, n(x)) = \lambda n_j(x)\partial_k + \mu n_k(x)\partial_j + \mu\delta_{jk}\partial_{n(x)}, \tag{1.5}$$

where $\partial_{n(x)}$ is the directional derivative along $n(x)$, i.e., $n_1(x)\partial_1 + n_2(x)\partial_2 + n_3(x)\partial_3$. Let $\nu(x)$ be the unit outward normal vector to S at a point $x \in S$. Further on, unless otherwise stated, we use $T = T(\partial_x, \nu(x))$ $(x \in S)$.

We consider four spectral problems with spectral parameter ζ; as we have mentioned above, all the other parameters are fixed constants.

I. Find the solutions $u(x)$ of the Lamé system (1.3) in G^+ with boundary condition

$$u^+ + \zeta T u^+ = 0 \quad \text{on } S. \tag{1.6}$$

II. Find the solutions $u(x)$ of the Lamé system (1.3) in G^- with radiation conditions at infinity and boundary condition

$$u^- = \zeta T u^- \quad \text{on } S. \tag{1.7}$$

III. Find the solutions $u(x)$ of the Lamé system (1.3) in $\mathbb{R}^3 \setminus S$ with radiation conditions at infinity and transmission conditions

$$u^+ = u^- \quad \text{and} \quad u^\pm = \zeta[Tu^- - Tu^+] \quad \text{on } S. \tag{1.8}$$

IV. Find the solutions $u(x)$ of the Lamé system (1.3) in $\mathbb{R}^3 \setminus S$ with radiation conditions at infinity and transmission conditions

$$Tu^+ = Tu^- \quad \text{and} \quad u^- - u^+ = \zeta T u^\pm \quad \text{on } S. \tag{1.9}$$

Let us formulate the *radiation conditions* for the solutions of (1.3) in G^- for $\omega \neq 0$ (cf. [22, Chapter 3, §2], and [8, §3.2]; in [22] ω is real). If $\omega \neq 0$, then any solution $u(x)$ admits the decomposition

$$u = u^{(p)} + u^{(s)}, \tag{1.10}$$

where $u^{(p)}$ and $u^{(s)}$ satisfy the equations

$$\begin{aligned} (\Delta + k_1^2)u^{(p)} &= 0, \qquad \operatorname{rot} u^{(p)} = 0, \\ (\Delta + k_2^2)u^{(s)} &= 0, \qquad \operatorname{div} u^{(s)} = 0 \end{aligned} \tag{1.11}$$

with
$$k_1 = \omega(\rho/(\lambda+2\mu))^{1/2}, \qquad k_2 = \omega(\rho/\mu)^{1/2}. \tag{1.12}$$
As $\lambda + \mu > 0$ according to (1.2), the numbers (1.12) are distinct. These potential and solenoidal components $u^{(p)}$ and $u^{(s)}$ are defined by the formulas
$$u^{(p)} = (k_2^2 - k_1^2)^{-1}(\Delta + k_2^2)u, \qquad u^{(s)} = (k_1^2 - k_2^2)^{-1}(\Delta + k_1^2)u. \tag{1.13}$$
We impose on $u^{(p)}$ and $u^{(s)}$ the standard radiation conditions for the solutions of the corresponding Helmholtz equations by requiring that
$$\partial_r u^{(p)}(x) - ik_1 u^{(p)}(x) = o(r^{-1}), \qquad \partial_r u^{(s)}(x) - ik_2 u^{(s)}(x) = o(r^{-1}) \tag{1.14}$$
as $r = |x| \to \infty$; here ∂_r is the directional derivative along the radius vector r of x. Equations (1.14) are the radiation conditions for the solutions of the system (1.3) for $\omega \neq 0$. One can observe from the integral representations given below (see Subsection 1.3) that $u^{(p)}$ and $u^{(s)}$ satisfy the following relations at infinity. If ω is real, then
$$u^{(p)}(x) = O(r^{-1}), \qquad \partial_r u^{(p)}(x) - ik_1 u^{(p)}(x) = O(r^{-2}), \tag{1.15}$$
$$u^{(s)}(x) = O(r^{-1}), \qquad \partial_r u^{(s)}(x) - ik_2 u^{(s)}(x) = O(r^{-2}) \tag{1.16}$$
as $r = |x| \to \infty$. Also (see [22, Chapter 3, §2]),
$$Tu^{(p)}(x) - ik_1(\lambda + 2\mu)u^{(p)}(x) = O(r^{-2}), \tag{1.17}$$
$$Tu^{(s)}(x) - ik_2\mu u^{(s)}(x) = O(r^{-2}), \tag{1.18}$$
$$u^{(p)}(x) \cdot u^{(s)}(x) = O(r^{-3}) \quad \text{and} \quad u^{(p)}(x) \cdot \overline{u}^{(s)}(x) = O(r^{-3}) \tag{1.19}$$
as $r = |x| \to \infty$; here $T = T(\partial, r_0(x))$, where $r_0(x) = r(x)/|r(x)|$.

If $\omega'' > 0$, then we can insert the exponential factors $\exp(-k_1'' r)$ and $\exp(-k_2'' r)$ into the right-hand sides of (1.15), (1.17) and (1.16), (1.18), respectively, where $k_j'' = \operatorname{Im} k_j$; thus, the right-hand sides decrease exponentially.

Finally, if $\omega = 0$, then, speaking about the radiation conditions, we actually assume that
$$u(x) = O(r^{-1}) \quad \text{and} \quad \partial_j u(x) = O(r^{-2}) \tag{1.20}$$
as $r = |x| \to \infty$. The term $O(r^{-2})$ in the second equation can be weakened to $o(r^{-1})$, but this implies the estimate $O(r^{-2})$.

1.2. *Surface Potentials*

We need to recall some well-known formulas and relations.

The *Kupradze matrix* for system (1.3) is the 3×3 matrix $\Gamma(x, \omega)$ with entries
$$\Gamma_{jk}(x, \omega) = \frac{\delta_{jk} e^{ik_2|x|}}{2\pi\mu|x|} + \frac{1}{2\pi\rho\omega^2}\,\partial_j\partial_k \frac{e^{ik_2|x|} - e^{ik_1|x|}}{|x|} \tag{1.21}$$
(see [22, Chapter 2, §1][1]). For $x \neq 0$, its rows and columns satisfy system (1.3). We have the estimates
$$|\Gamma_{jk}(x, \omega)| \leqslant C|x|^{-1} \tag{1.22}$$

[1]We recall once more that in [22] ω is assumed to be real. Here and below the extensions of results to complex ω are obvious.

uniformly over $|x| \leqslant \text{const}$, $|\omega| \leqslant \text{const}$.

As $\omega \to 0$, $x \neq 0$, the matrix $\Gamma(x, \omega)$ converges to the *Kelvin matrix* with entries

$$\Gamma_{jk}(x) = \lambda' \frac{\delta_{jk}}{|x|} + \mu' \frac{x_j x_k}{|x|^3}, \tag{1.23}$$

where

$$\lambda' = \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)}, \qquad \mu' = \frac{\lambda + \mu}{4\pi\mu(\lambda + 2\mu)}; \tag{1.24}$$

for $x \neq 0$, its rows and columns satisfy system (1.3) with $\omega = 0$. For the entries of the difference

$$\overset{\circ}{\Gamma}(x, \omega) = \Gamma(x, \omega) - \Gamma(x), \tag{1.25}$$

one has the estimates (see [22, Chapter 2, §1])

$$|\overset{\circ}{\Gamma}_{jk}(x, \omega)| \leqslant C|\omega|, \qquad |\partial_l \overset{\circ}{\Gamma}_{jk}(x, \omega)| \leqslant C|\omega|^2, \tag{1.26}$$

which are uniform over $|x| \leqslant \text{const}$, $|\omega| \leqslant \text{const}$. The second derivatives of $\overset{\circ}{\Gamma}(x, \omega)$ have isolated singularities of the order of $|x|^{-1}$ at the origin. The entries of the Kupradze matrix satisfy the radiation conditions (1.14) and consequently the stronger conditions (1.15)–(1.19), and the entries of the Kelvin matrix satisfy the estimates (1.20). Furthermore, if F is, say, a compactly supported continuous function, then the volume potential

$$U(x) = -\frac{1}{2} \int \Gamma(x, \omega) F(y)\, dy \tag{1.27}$$

satisfies the equation $L(\partial_x, \omega)U = F$, so that

$$\mathcal{E}(x, \omega) = -\tfrac{1}{2} \Gamma(x, \omega) \tag{1.28}$$

is a fundamental solution for the operator $L(\partial_x, \omega)$. When $\omega = 0$, the same is true of $\mathcal{E}(x) = -\frac{1}{2}\Gamma(x)$ and $L(\partial_x)$ (see [22, Chapter 5, §10]).

Let us introduce the surface potentials: the single layer potential

$$\mathcal{A}(\omega)\varphi(x) = \int_S \mathcal{E}(x, \omega)\varphi(y)\, dS_y \qquad (x \in \mathbb{R}^3) \tag{1.29}$$

and the double layer potential

$$\mathcal{B}(\omega)\psi(x) = \int_S [T(\partial_y, \nu(y))\mathcal{E}(x - y, \omega)]' \psi(y)\, dS_y \qquad (x \notin S). \tag{1.30}$$

Here and below the prime $'$ denotes a transposed matrix. We shall make more precise assumptions concerning the densities φ and ψ later; for the time being we assume that they are infinitely smooth.

Let us also introduce the following integral operators on S:

$$A(\omega)\varphi(x) = \int_S \mathcal{E}(x - y, \omega)\varphi(y)\, dS_y \qquad (x \in S), \tag{1.31}$$

$$B(\omega)\psi(x) = \int_S [T(\partial_y, \nu(y))\mathcal{E}(x - y, \omega)]' \psi(y)\, dS_y \qquad (x \in S). \tag{1.32}$$

For $\omega = 0$, the matrix $\mathcal{E}(x,\omega)$ must be replaced by $\mathcal{E}(x)$, and we shall write $\mathcal{A}$, $\mathcal{B}$, A, and B instead of $\mathcal{A}(0)$, $\mathcal{B}(0)$, $A(0)$, and $B(0)$, respectively.

¿From the structure of the kernel in (1.31) it follows that $A(\omega)$ is a classical, or polyhomogeneous, ΨDO of order -1 (cf. [28] or [2, Section 1.6]). Therefore it is bounded as an operator from $H^t(S)$ into $H^{t+1}(S)$ for all t. It is well known (see [22, Chapter 5]) that the potential $\mathcal{A}(\omega)\varphi(x)$ has the boundary values

$$\mathcal{A}(\omega)\varphi^{\pm} = A(\omega)\varphi. \tag{1.33$^\pm$}$$

If $\varphi \in H^t(S)$, then $\mathcal{A}(\omega)\varphi$ is the solution of the Dirichlet problem with the Dirichlet data from $H^{t+1}(S)$. This boundary value problem is elliptic (see below), and it is easy to conclude either from this fact or from the structure of the kernel of the operator $\mathcal{A}(\omega)$ that it is bounded as an operator acting from $H^t(S)$ into $H^{t+3/2}(G^+)$ and $H^{t+3/2}_{\rm loc}(G^-)$. Here and below "loc" can be omitted for $\omega'' > 0$.

The first derivatives of $\mathcal{A}\varphi(x)$ have the boundary values in $H^0(S)$ (cf. [22, Chapter 5]). In particular,

$$T\mathcal{A}(\omega)\varphi^{\pm} = \mp\tfrac{1}{2}\varphi + B'(\omega)\varphi. \tag{1.34$^\pm$}$$

Here

$$B'(\omega)\varphi(x) = \int_S [T(\partial_x, \nu(x))\mathcal{E}(x-y,\omega)]\varphi(y)\,dS_y \tag{1.35}$$

is the transpose of $B(\omega)$:

$$\int_S B(\omega)\varphi\cdot\psi\,dS = \int_S \varphi\cdot B'(\omega)\psi\,dS \qquad (\varphi,\psi\in H^0(S)). \tag{1.36}$$

Both operators $B(\omega)$ and $B'(\omega)$ are singular integral operators (ΨDOs of order 0), and the integrals in (1.32) and (1.35) must be understood in the sense of the Cauchy principal value. These operators are bounded in all spaces $H^t(S)$. The potential $\mathcal{B}(\omega)$ is a bounded operator from $H^t(S)$ into $H^{t+1/2}(G^-)$ and $H^{t+1/2}_{\rm loc}(G^-)$ for $t \geqslant 0$. The boundary values of $\mathcal{B}(\omega)\psi$ exist and satisfy the relations

$$\mathcal{B}(\omega)\psi^{\pm} = \pm\tfrac{1}{2}\psi + B(\omega)\psi. \tag{1.37$^\pm$}$$

Formulas (1.33$^\pm$), (1.34$^\pm$), and (1.37$^\pm$) can be extended to φ, $\psi \in H^0(S)$ by passing to the limits (cf. [29] and Section 2). Note also that for $\psi \in H^1(S)$ one has

$$T\mathcal{B}(\omega)\psi^+ = T\mathcal{B}(\omega)\psi^-. \tag{1.38}$$

The proof of this formula will be recalled in Subsection 1.7.

1.3. *Integral Formulas*

We have to recall Green's formulas for the Lamé operator in G^+, which can be obtained by integration by parts (cf. [22, Chapter 3, §1]). The first formula

$$\int_{G^+} L(\partial_x)u\cdot v\,dx = -\int_{G^+} E(u,v)\,dx + \int_S Tu^+\cdot v^+ dS \tag{1.39}$$

holds, say, for $u \in H^2(G^+)$ and $v \in H^1(G^+)$; here and in what follows

$$E(u,v) = \lambda\,\mathrm{div}\,u\cdot\mathrm{div}\,v + \mu\sum(\partial_q u_p + \partial_p u_q)\partial_q v_p\,. \tag{1.40}$$

Formula (1.39) implies Green's second formula

$$\int_{G^+}(L(\partial_x)u\cdot v - u\cdot L(\partial_x)v)\,dx = \int_S (Tu^+\cdot v^+ - u^+\cdot Tv^+)\,dS \tag{1.41}$$

for $u, v \in H^2(S)$. The connectedness of the boundary is here unessential.

Let us note some obvious corollaries. If u is a solution of the homogeneous system (1.3), then

$$-\rho\omega^2 \int_{G^+} u \cdot v \, dx = -\int_{G^+} E(u,v)\, dx + \int_S Tu^+ \cdot v^+ dS, \tag{1.42}$$

and if u and v are solutions of system (1.3) with, generally, different $\omega = \omega_1$ and $\omega = \omega_2$, then

$$\rho(\omega_2^2 - \omega_1^2) \int_{G^+} u \cdot v \, dx = \int_S (Tu^+ \cdot v^+ - u^+ \cdot Tv^+)\, dx. \tag{1.43}$$

Formula (1.42) is valid for a solution u of (1.3) from $H^{3/2}(G^+)$ and a function $v \in H^1(G^+)$, and formula (1.43) is valid for solutions from $H^{3/2}(G^+)$ (cf. Subsection 2.3; inside the domain the solutions are infinitely differentiable).

We can obtain an integral representation for solutions of system (1.3) belonging to $H^{3/2}(G^+)$ in the usual way from formula (1.43) (cf. [22, Chapter 3, §2]):

$$u(x) = \mathcal{B}(\omega)u^+(x) - \mathcal{A}(\omega)Tu^+(x). \tag{1.44}$$

When $x \in G^-$, the right-hand side in (1.44) vanishes. For solutions in G^- belonging to $H^{3/2}_{\mathrm{loc}}(G^-)$ and satisfying the radiation conditions, one obtains

$$u(x) = \mathcal{A}(\omega)Tu^-(x) - \mathcal{B}(\omega)u^-(x), \tag{1.45}$$

and the right-hand side vanishes for $x \in G^+$.

Passing to the limit as $x \to S$ in (1.44) and (1.45) and using ($1.33^\pm$) and ($1.37^\pm$), we obtain the relations

$$(B(\omega) \mp \tfrac{1}{2}I)u^\pm = A(\omega)Tu^\pm \tag{$1.46^\pm$}$$

on S. Similar formulas hold for $\omega = 0$.

1.4. *Reduction of Problems* I–IV *to Integral Equations on S for Nonexceptional ω*

If the operator $\frac{1}{2}I - B(\omega)$ is invertible, let us set

$$T^+(\omega) = (\tfrac{1}{2}I - B(\omega))^{-1}A(\omega). \tag{1.47}$$

Then (1.46^+) implies

$$u^+ = -T^+(\omega)Tu^+, \tag{1.48}$$

and Problem I is reduced by the substitution

$$\varphi = Tu^+ \tag{1.49}$$

to the equation

$$T^+(\omega)\varphi = \zeta\varphi \quad \text{on } S. \tag{1.50}$$

Likewise, if the operator $\frac{1}{2}I + B(\omega)$ is invertible, we set

$$T^-(\omega) = (\tfrac{1}{2}I + B(\omega))^{-1}A(\omega). \tag{1.51}$$

Then (1.46^-) implies

$$u^- = T^-(\omega)Tu^-, \tag{1.52}$$

and Problem II is reduced by the substitution

$$\varphi = Tu^- \tag{1.53}$$

to the equation

$$T^-(\omega)\varphi = \zeta\varphi \quad \text{on } S. \tag{1.54}$$

Furthermore, if $u^+ = u^-$, then from $(1.46^{\pm})$ we obtain

$$u = A(\omega)(Tu^- - Tu^+), \tag{1.55}$$

and Problem III is reduced by the substitution

$$\varphi = Tu^- - Tu^+ \tag{1.56}$$

to the equation

$$A(\omega)\varphi = \zeta\varphi \quad \text{on } S. \tag{1.57}$$

Finally, if both operators $\frac{1}{2}I \pm B(\omega)$ are invertible and $Tu^+ = Tu^-$, then from (1.48) and (1.52) we obtain

$$u^- - u^+ = (T^+(\omega) + T^-(\omega))Tu^{\pm}, \tag{1.58}$$

and Problem IV is reduced by the substitution

$$\varphi = Tu^{\pm} \tag{1.59}$$

to the equation

$$T(\omega)\varphi = \zeta\varphi \quad \text{on } S, \tag{1.60}$$

where

$$T(\omega) = T^+(\omega) + T^-(\omega) = (\tfrac{1}{4}I - B^2(\omega))^{-1}A(\omega); \tag{1.61}$$

see also formula (1.107) below.

The values of ω for which at least one of the operators $\frac{1}{2}I \pm B(\omega)$ is not invertible in $H^0(S)$ will be called *exceptional*. We will relate the exceptional ω with the nonuniqueness for the first and the second boundary value problems of elasticity at the end of Subsection 1.6. We did not assume that ω is nonexceptional when reducing Problem III to equation (1.57).

Obviously, the operators in the equations obtained above can be nonselfadjoint in $L_2(S)$ (see Subsections 1.8–1.10), and then they can have associated functions. All the problems and equations we have indicated relate to the eigenfunctions only. We avoid writing out the equations for the associated functions and concentrate on the study of spectral properties of the operators $T^{\pm}(\omega)$, $A(\omega)$, and $T(\omega)$.

The problems of finding eigenfunctions for nonexceptional ω are equivalent to the equations stated above. We shall comment on this in Subsection 1.7.

1.5. *The First and the Second Boundary Value Problems. Eigenvalues; Uniqueness Theorems*

The interior and exterior *first* (or *Dirichlet*) boundary value problems for system (1.3) in $G^{\pm}$ with the boundary condition

$$u^{\pm} = f \quad \text{on } S, \tag{$1.62^{\pm}$}$$

respectively, will be denoted by $D^+(\omega)$ and $D^-(\omega)$.

The interior and exterior *second* (or *Traction*, or *Neumann*) boundary value problems for system (1.3) in $G^{\pm}$ with the boundary condition

$$Tu^{\pm} = g \quad \text{on } S, \tag{$1.63^{\pm}$}$$

respectively, will be denoted by $N^+(\omega)$ and $N^-(\omega)$.

Of course, in the case of exterior problems the radiation conditions must be imposed.

Let us briefly recall some properties of these boundary value problems; we shall need these details later on. The Lamé operator $L(\partial_x)$ is a formally selfadjoint homogeneous second order operator with constant coefficients. It has the symbol

$$l(\xi) = -|\xi|^2(\mu E + (\lambda+\mu)\Lambda(\xi)), \tag{1.64}$$

where $\Lambda(\xi)$ is the matrix with entries $\xi_j\xi_k|\xi|^{-2}$ and E is the identity matrix. It is easy to check that

$$\Lambda^2(\xi) = \Lambda(\xi). \tag{1.65}$$

It follows that $l(\xi)$ has the inverse matrix

$$l^{-1}(\xi) = -\frac{1}{\mu|\xi|^2}\left(E - \frac{\lambda+\mu}{\lambda+2\mu}\Lambda(\xi)\right). \tag{1.66}$$

This confirms the ellipticity of the Lamé operator. Its formal selfadjointness is obvious. The symbol $-l(\xi)$ is a positive definite matrix: if v is a vector from $\mathbb{C}^3$, then

$$-l(\xi)v\cdot\overline{v} = \mu|\xi|^2|v|^2 + (\lambda+\mu)|\xi\cdot v|^2 \geqslant \mu|\xi|^2|v|^2. \tag{1.67}$$

This Hadamard–Legendre condition implies, in particular, that the operator $-L(\partial_x)$ is strongly elliptic. In turn, this implies the ellipticity of the problems $D^\pm(\omega)$. The problems $N^\pm(\omega)$ are also elliptic, which can be established directly by checking the corresponding Shapiro–Lopatinskii condition (see the English edition of [22, Chapter 1, §15]).

The natural function spaces for these problems are obvious:

For the problems $D^\pm(\omega)$: $f \in H^t(S)$, $u \in H^{t+1/2}(G^+)$ or $u \in H^{t+1/2}_{\mathrm{loc}}(G^-)$.

For the problems $N^\pm(\omega)$: $g \in H^t(S)$, $u \in H^{t+3/2}(G^+)$ or $u \in H^{t+3/2}_{\mathrm{loc}}(G^-)$.

In all cases we shall only consider the values $t \geqslant 0$; “loc” can be omitted if $\omega'' > 0$. Additionally, the function u is assumed to be infinitely smooth in $G^\pm$ and to satisfy the radiation conditions in G^-.

We shall also use another statements of the interior problems:

$$L(\partial_x,\omega)u = F \quad \text{in } G^+, \qquad u^+ = 0 \quad \text{on } S, \tag{1.68}$$

$$L(\partial_x,\omega)u = F \quad \text{in } G^+, \qquad Tu^+ = 0 \quad \text{on } S. \tag{1.69}$$

When $F = 0$, (1.68) and (1.69) are the spectral boundary value problems for $-L(\partial_x)$ with spectral parameter $\eta = \rho\omega^2$.

Let us introduce the operator L_D in $L_2(G^+)$ acting as $L(\partial_x)$ with domain

$$\mathcal{D}(L_D) = \{u \in H^2(G^+) : u^+ = 0\} = H^2(G^+) \cap H^1_0(G^+), \tag{1.70}$$

where $H^1_0(G^+)$ is the closure of $C^\infty_0(G^+)$ in $H^1(G^+)$. It is well known that L_D is a closed densely defined selfadjoint operator with compact resolvent. Moreover, $-L_D$ is positive by virtue of the Gårding inequality

$$\varepsilon\|u\|^2_{1,G^+} \leqslant -(L_D u, u)_{G^+} \qquad (u \in \mathcal{D}(L_D)) \tag{1.71}$$

with a positive ε. In turn, the Gårding inequality can be obtained, e.g., using the Friedrichs inequality

$$\|u\|_{0,G^+} \leqslant C\|\operatorname{grad} u\|_{0,G^+} \qquad (u \in H^1_0(G^+)), \tag{1.72}$$

Korn's first inequality (cf., e.g., [27, Chapter 1])

$$\|\operatorname{grad} u\|^2_{0,G^+} \leqslant \frac{1}{2}\int_{G^+}\sum |\partial_q u_p + \partial_p u_q|^2\,dx \qquad (u \in H^1_0(G^+)), \tag{1.73}$$

Green's formula (1.39) and the identity (see [22, Chapter 3, §1])

$$E(u,\overline{u}) = \frac{3\lambda+2\mu}{3}|\operatorname{div} u|^2 + \frac{\mu}{2}\sum_{p\neq q}|\partial_q u_p + \partial_p u_q|^2 + \frac{\mu}{3}\sum_{p,q}|\partial_p u_p - \partial_q u_q|^2, \tag{1.74}$$

in which the right-hand side majorizes the integrand on the right-hand side in (1.73) in view of the elementary inequality $3|a| \leqslant |a+b+c| + |a-b| + |a-c|$.

We also introduce the operator L_N in $L_2(G^+)$ acting as $L(\partial_x)$ with domain

$$\mathcal{D}(L_N) = \{u \in H^2(G) : Tu^+ = 0\}. \tag{1.75}$$

This is again a closed densely defined selfadjoint operator with compact resolvent. This operator is nonnegative by virtue of Green's formula (1.39) and identity (1.74). One can obtain the analog of (1.71) in the form

$$\varepsilon\|u\|^2_{1,G^+} \leqslant -(L_N u, u)_{G^+} + \|u\|^2_{0,G^+} \qquad (u \in \mathcal{D}(L_N)), \tag{1.76}$$

using, e.g., Green's formula (1.39), identity (1.74), and Korn's second inequality (cf. [27, Chapter 1]) for functions $u \in H^1(G^+)$:

$$\|\operatorname{grad} u\|^2_{0,G^+} \leqslant C\biggl(\int_{G^+}\sum |\partial_q u_p + \partial_p u_q|^2\,dx + \|u\|^2_{0,G^+}\biggr). \tag{1.77}$$

(Inequalities (1.71) and (1.76) can also be obtained by means of the Fourier transform.)

Thus, the spectrum of $-L_D$ consists of isolated positive eigenvalues of finite multiplicity. Numbering them in nondecreasing order with the account of multiplicities, we obtain a sequence $\{\eta_j(-L_D)\}_1^\infty$ with the asymptotics $\eta_j(-L_D) \sim \text{const}\ j^{2/3}$. Similarly, the spectrum of $-L_N$ consists of isolated nonnegative eigenvalues of finite multiplicities, and the corresponding sequence $\{\eta_j(-L_N)\}_1^\infty$ has the same asymptotics $\eta_j(-L_N) \sim \text{const}\ j^{2/3}$. See [35].

The eigenfunctions of the operators L_D and L_N belong to $C^\infty(\overline{G}^+)$ in view of the ellipticity of the problems (1.68) and (1.69) and the infinite smoothness of the boundary.

Obviously, the homogeneous forms of the problems (1.68) and $D^+(\omega)$ coincide, and therefore the uniqueness for $D^+(\omega)$ occurs if and only if $\rho\omega^2$ differs from all eigenvalues of the operator $-L_D$. In particular, we have the uniqueness for all nonreal ω and $\omega = 0$. Similarly, we have the uniqueness for $N^+(\omega)$ if and only if $\rho\omega^2$ differs from all eigenvalues of the operator $-L_N$. In particular, we have the uniqueness for all nonreal ω.

For the exterior problems $D^-(\omega)$ and $N^-(\omega)$, the uniqueness holds for all ω (with $\omega'' \geqslant 0$). This can be checked using a formula of the type (1.42) for the domain

$$G^-_R = G^- \cap E_R, \quad \text{where } E_R = \{x : |x| < R\} \tag{1.78}$$

with R such that $\overline{G}^+ \subset E_R$:

$$-\rho\omega^2\int_{G^-_R}|u|^2\,dx = -\int_{G^-_R}E(u,\overline{u})\,dx - \int_S Tu^-\cdot\overline{u}^-\,dS + \int_{S_R}Tu\cdot\overline{u}\,dS, \tag{1.79}$$

where $S_R = \{x : |x| = R\}$. The proof of the uniqueness for real values of ω can be found in [22, Chapter 3, §2]. We shall use similar reasoning when proving that our operators are dissipative (see Subsections 1.8–1.9 below). If ω is nonreal, then, taking the limit as $R \to \infty$ in (1.79) for the solution with $u^- = 0$ or $Tu^- = 0$, we get

$$\rho\omega^2\int_{G^-}|u|^2\,dx = \int_{G^-}E(u,\overline{u})\,dx, \tag{1.80}$$

since the integral over the sphere S_R tends to zero. It follows that here both integrals vanish for pure imaginary ω (when they are of different sign) as well as for not pure imaginary ω (when ω^2 is not real). Thus, the only solution of the homogeneous problems is $u \equiv 0$.

1.6. *The Operators $A(\omega)$ and $\frac{1}{2}I \pm B(\omega)$ and Their Invertibility*

We start with the following

Proposition 1.1. *The principal symbol of $A(\omega)$ is a Hermitian negative definite matrix. In particular, $A(\omega)$ is an elliptic pseudodifferential operator of order -1, it is Fredholm as an operator from $H^t(S)$ into $H^{t+1}(S)$ and has index zero.*

Proof. We compute the principal symbol using a lemma from [3]. Let x_0 be a point of S. We shift the origin of the coordinate system in $\mathbb{R}^3$ to x_0 and rotate the axes in such a way that x_3 takes the direction of the exterior (for definiteness) normal to S at this point. Let $x_3 = X(x') = X(x_1, x_2)$ be the equation of S near x_0. We define the new coordinates $\tilde{x}_j$ $(j = 1, 2, 3)$ rectifying S:

$$\tilde{x}_1 = x_1, \qquad \tilde{x}_2 = x_2, \qquad \tilde{x}_3 = x_3 - X(x'). \tag{1.81}$$

For simplicity, below we omit the tildes in the notation.

The complete and the principal symbol of the operator (1.27) in the old coordinates is $l^{-1}(\xi)$ (see (1.66)). Dividing it by 2π and integrating over the ξ_3-axis, we obtain the principal symbol $\sigma_A(0, \xi')$ of the operator A in the new coordinates at the point $x' = 0$:

$$\sigma_A(0, \xi') = \frac{1}{2\pi} \int_{-\infty}^{\infty} l^{-1}(\xi)\, d\xi_3 = \frac{1}{2\pi\mu} \int_{-\infty}^{\infty} \left(\frac{\lambda + \mu}{\lambda + 2\mu} \frac{1}{|\xi|^2} \Lambda(\xi) - \frac{1}{|\xi|^2} E \right) d\xi_3,$$

where $\xi' = (\xi_1, \xi_2)$. Using the easily verified relations

$$\int_{-\infty}^{\infty} \frac{1}{|\xi|^2}\, d\xi_3 = \frac{\pi}{|\xi'|}, \qquad \int_{-\infty}^{\infty} \frac{\xi_3^2}{|\xi|^4}\, d\xi_3 = \frac{\pi}{2|\xi'|}, \qquad \int_{-\infty}^{\infty} \frac{1}{|\xi|^4}\, d\xi_3 = \frac{\pi}{2|\xi'|^3},$$

we obtain

$$\sigma_A(0, \xi') = \frac{1}{2\mu|\xi'|} \left(\frac{\lambda + \mu}{2(\lambda + 2\mu)} \begin{pmatrix} \Lambda(\xi') & 0 \\ 0 & 1 \end{pmatrix} - E \right). \tag{1.82}$$

This matrix is Hermitian and has the negative eigenvalues

$$-\frac{1}{2\mu|\xi'|} \quad \text{and} \quad (\text{double}) \quad -\frac{\lambda + 3\mu}{4(\lambda + 2\mu)\mu|\xi'|}\,. \tag{1.83}$$

This implies, in particular, that A is elliptic, so that $A(\omega)$ is Fredholm as an operator acting from $H^t(S)$ into $H^{t+1}(S)$, with index independent of t. Since the principal symbol is Hermitian, the index is 0 for all t.

Proposition 1.2. *The operator $A(\omega)\colon H^t(S) \to H^{t+1}(S)$ is invertible if and only if the homogeneous problem $D^+(\omega)$ has no nontrivial solutions. For other values of ω,*

$$\operatorname{Ker} A(\omega) = \{Tu^+ : \ L(\partial_x, \omega)u = 0 \ \textit{ in } \ G^+,\ u^+ = 0\}. \tag{1.84}$$

In particular, the dimension of the null space (1.84) *is equal to the dimension of the space of solutions of the homogeneous problem $D^+(\omega)$ and to the dimension of the eigenspace of the operator $-L_D$ corresponding to the eigenvalue $\rho\omega^2$.*

Proof. Let u be a nontrivial solution of the problem $D^+(\omega)$. Then (1.46^+) implies that $A(\omega)Tu^+ = 0$.

Conversely, let $A(\omega)\varphi = 0$. We set $u = -\mathcal{A}(\omega)\varphi$. Then $u^\pm = 0$, and since the solution of the exterior problem $D^-(\omega)$ is unique, we have $u = 0$ in G^- and $Tu^- = 0$. Now ($1.34^\pm$) gives $Tu^+ = \varphi$. It remains to note that u^+ and Tu^+ can vanish simultaneously if and only if the solution u of the homogeneous problem $D^+(\omega)$ is trivial, which can be seen, e.g., from (1.44).

Proposition 1.3. *The operators* $\frac{1}{2}I \pm B(\omega)$ *have Hermitian principal symbols with positive determinants, so they are elliptic* ΨDOs *of order zero and Fredholm operators of index zero in all* $H^t(S)$.

Proof. It is shown in [22, Chapter 2, §4] that

$$(T(\partial_x, n(x))\Gamma(x))_{jk} = \mu(\lambda' - \mu')\frac{n_j(x)x_k - n_k(x)x_j}{|x|^3} + \left(\mu(\mu' - \lambda')\delta_{jk} - 6\mu\mu'\frac{x_j x_k}{|x|^2}\right)\sum_{l=1}^{3} n_l(x)\frac{x_l}{|x|^3}. \tag{1.85}$$

We also have the estimates (1.26) for $\overset{\circ}{\Gamma}(x,\omega) = \Gamma(x,\omega) - \Gamma(x)$. Since the normal vector is orthogonal to tangential vectors, on the surface S we have

$$\left|\sum_l \nu_l(y)\frac{x_l - y_l}{|x-y|}\right| \leqslant C|x-y|. \tag{1.86}$$

In view of (1.28), the principal singular part of the kernel of the operator $B(\omega)$ has the entries

$$\mu(\lambda' - \mu')\frac{\nu_k(y)(x_j - y_j) - \nu_j(y)(x_k - y_k)}{2\,|x-y|^3}. \tag{1.87}$$

Here, according to (1.24),

$$\lambda' - \mu' = \frac{1}{2\pi(\lambda + 2\mu)} > 0. \tag{1.88}$$

Let us now use the same coordinate system as in the calculation of the symbol of $A(\omega)$. Since

$$\frac{x_j}{|x'|^3} = -\partial_j \frac{1}{|x'|} \qquad (j = 1, 2)$$

and the kernel $(2\pi|x' - y'|)^{-1}$ corresponds to the principal symbol $|\xi'|^{-1}$ (see the calculation of the principal symbol of an operator of the type A for the Laplace equation, e.g., in [1, §36]), we see that the principal symbol of $B(\omega)$ is

$$\sigma_B(\xi') = \frac{\pi\mu(\lambda' - \mu')i}{|\xi'|}\begin{pmatrix} 0 & 0 & -\xi_1 \\ 0 & 0 & -\xi_2 \\ \xi_1 & \xi_2 & 0 \end{pmatrix}. \tag{1.89}$$

It follows that

$$\det(\tfrac{1}{2}E \pm \sigma_B(\xi')) = \frac{1}{8}\left(1 - \frac{\mu^2}{(\lambda + 2\mu)^2}\right) > 0 \tag{1.90}$$

(cf. [22, Chapter 6, §3]). Hence $\frac{1}{2}I \pm B(\omega)$ are elliptic operators of order zero, and they have index zero since their principal symbols are Hermitian.

Corollary 1.4. *The assertion of Proposition* 1.3 *remains true for the operators* $\frac{1}{2}I \pm B^*(\omega)$ *and* $\frac{1}{2}I \pm B'(\omega)$.

Proposition 1.5. *The operator* $\frac{1}{2}I + B'(\omega)$ *in* $H^t(S)$ *is invertible if and only if the homogeneous problem* $D^+(\omega)$ *has no nontrivial solutions. For other values of* ω,

$$\operatorname{Ker}(\tfrac{1}{2}I + B'(\omega)) = \{Tu^+ : L(\partial_x, \omega)u = 0 \ \text{ in } G^+,\ u^+ = 0\} = \operatorname{Ker} A(\omega). \tag{1.91}$$

Proof. Since $\operatorname{ind}(\frac{1}{2}I + B') = 0$, it suffices to prove the first equality in (1.91), but it is already known: see [22, Chapter 7, §2].

Obviously, the operators $\frac{1}{2}I + B(\omega)$ and $\frac{1}{2}I + B^*(\omega)$ are invertible in $H^t(S)$ whenever $\frac{1}{2}I + B'(\omega)$ is invertible.

Proposition 1.6. *The operator* $\frac{1}{2}I - B(\omega)$ *in* $H^t(S)$ *is invertible if and only if the homogeneous problem* $N^+(\omega)$ *has no nontrivial solutions. For other values of* ω,

$$\operatorname{Ker}(\tfrac{1}{2}I - B(\omega)) = \{u^+ : L(\partial_x, \omega)u = 0 \ \text{ in } \ G^+, \, Tu^+ = 0\}, \tag{1.92}$$

so that the dimension of the null space (1.92) *is equal to the dimension of the space of solutions of the homogeneous problem* $N^+(\omega)$ *and to the dimension of the eigenspace of the operator* $-L_N$ *corresponding to the eigenvalue* $\rho\omega^2$.

Proof. Since $\operatorname{ind}(\frac{1}{2}I - B) = 0$, it suffices to note that (1.92) is proved in [22, Chapter 7, §2]. See also Subsection 2.5, where we check (1.92) for Lipschitz domains.

The operators $\frac{1}{2}I - B'(\omega)$ and $\frac{1}{2}I - B^*(\omega)$ are invertible in $H^t(S)$ whenever $\frac{1}{2}I - B(\omega)$ is invertible.

Remark 1.7. The spaces $\operatorname{Ker} A(\omega)$, $\operatorname{Ker}(\frac{1}{2}I - B(\omega))$, and $\operatorname{Ker}(\frac{1}{2}I + B'(\omega))$ coincide with their complex conjugates, i.e., are invariant with respect to the mapping $\varphi \mapsto \overline{\varphi}$.

This follows from (1.84), (1.91), and (1.92), since all exceptional values of ω are real.

Making the definition of exceptional ω given at the end of Subsection 1.4 more precise, we will call a frequency ω *exceptional with respect to the problem* D^+ if $\rho\omega^2 = \eta_j(-L_D)$ for some j, and *exceptional with respect to the problem* N^+ if $\rho\omega^2 = \eta_j(-L_N)$ for some j. The fact that ω is not exceptional with respect to the problem D^+ is equivalent to the invertibility of the operators $A(\omega)$, $\frac{1}{2}I + B(\omega)$, and $\frac{1}{2}I + B'(\omega)$, and the fact that ω is not exceptional with respect to the problem N^+ is equivalent to the invertibility of the operators $\frac{1}{2}I - B(\omega)$ and $\frac{1}{2}I - B'(\omega)$.

1.7. *Formulas for Solutions of the Main Boundary Value Problems and Some Other Equalities*

We need to recall how to construct the solutions of the main boundary value problems for nonexceptional ω using potentials (cf. [22, Chapter 7]).

Let ω be nonexceptional with respect to the problem D^+. Then the solution of the nonhomogeneous problem $D^+(\omega)$ can be constructed as a double layer potential $u = \mathcal{B}(\omega)\psi$. Formula (1.37$^+$) gives

$$u = \mathcal{B}(\omega)(\tfrac{1}{2}I + B(\omega))^{-1}u^+. \tag{1.93}$$

Another possibility is to seek the solution as a single layer potential $u = \mathcal{A}(\omega)\varphi$, which leads to

$$u = \mathcal{A}(\omega)(A(\omega))^{-1}u^+. \tag{1.94}$$

A similar formula gives the solution of the problem $D^-(\omega)$:

$$u = \mathcal{A}(\omega)(A(\omega))^{-1}u^-. \tag{1.95}$$

Also, we can seek the solution of the problem $N^-(\omega)$ as a single layer potential $u = \mathcal{A}(\omega)\varphi$; formula (1.34$^-$) gives

$$u = \mathcal{A}(\omega)(\tfrac{1}{2}I + B'(\omega))^{-1}Tu^-. \tag{1.96}$$

Now let ω be nonexceptional with respect to the problem N^+. Then the solution of the problem $N^+(\omega)$ can be constructed as a single layer potential $u = \mathcal{A}(\omega)\varphi$. By (1.34$^+$), we obtain

$$u = -\mathcal{A}(\omega)(\tfrac{1}{2}I - B'(\omega))^{-1}Tu^+. \tag{1.97}$$

Also, the solution of the problem $D^-(\omega)$ can be constructed as a double layer potential $u = \mathcal{B}(\omega)\psi$. By (1.37$^-$), we obtain

$$u = -\mathcal{B}(\omega)(\tfrac{1}{2}I - B(\omega))^{-1}u^-. \tag{1.98}$$

To complete the picture, let us recall that the method of integral equations allows one to construct the solutions of the main exterior and (under corresponding orthogonality conditions) interior boundary value problems for exceptional ω as well, see [22, Chapter 7]; cf. also Subsection 1.11 below.

Now let us comment on the equivalence of Problems I–IV for the eigenfunctions to the corresponding equations obtained in Subsection 1.4 (for ω such that these equations make sense). If φ is a solution of (1.50), then the corresponding solution of Problem I is reconstructed as the solution of the problem $N^+(\omega)$ with $Tu^+ = \varphi$. If φ is a solution of (1.54), then the corresponding solution of Problem II is reconstructed as the solution of the problem $N^-(\omega)$ with $Tu^- = \varphi$. If φ is a solution of (1.57), then the corresponding solution of Problem III is reconstructed as $u = \mathcal{A}(\omega)\varphi$. Finally, if φ is a solution of (1.60), then the corresponding solution of Problem IV is reconstructed as the solution of the problems $N^\pm(\omega)$ with $Tu^\pm = \varphi$. It is easy to see that in all the cases the correspondence is bijective. The question of smoothness is solved trivially: all the eigenfunctions (and in general all the root functions) are infinitely smooth due to the ellipticity of boundary value problems and operators under consideration and the infinite smoothness of S.

We shall also need the following two statements (cf. [13, p. 89], for more general systems).

Proposition 1.8. *For all* ω,

$$B(\omega)A(\omega) = A(\omega)B'(\omega). \tag{1.99}$$

Proof. Let ω be nonreal, so that the operator $A(\omega)$ is invertible (see Proposition 1.2). From (1.48) and (1.47) we have

$$Tu^+ = -A^{-1}(\omega)(\tfrac{1}{2}I - B(\omega))u^+. \tag{1.100}$$

On the other hand, for $u = \mathcal{A}(\omega)\varphi$ we have

$$u^+ = A(\omega)\varphi \quad \text{and} \quad Tu^+ = -(\tfrac{1}{2}I - B'(\omega))\varphi$$

(see (1.34^+)), and therefore,

$$Tu^+ = -(\tfrac{1}{2}I - B'(\omega))A^{-1}(\omega)u^+. \tag{1.101}$$

Comparing (1.100) with (1.101), we obtain, since u^+ is arbitrary,

$$(\tfrac{1}{2}I - B(\omega))A(\omega) = A(\omega)(\tfrac{1}{2}I - B'(\omega)). \tag{1.102}$$

Instead, we can now construct two representations of Tu^- in terms of u^- and obtain

$$(\tfrac{1}{2}I + B(\omega))A(\omega) = A(\omega)(\tfrac{1}{2}I + B'(\omega)). \tag{1.103}$$

Both (1.102) and (1.103) imply (1.99). This formula can now be extended to real ω by a passage to the limit.

Proposition 1.9. *Let* ω *be nonexceptional with respect to the problem* D^+. *Then*

$$T\mathcal{B}(\omega)\psi^+ = T\mathcal{B}(\omega)\psi^- = A^{-1}(\omega)(B^2(\omega) - \tfrac{1}{4}I)\psi. \tag{1.104}$$

The first of these equalities is valid for all ω.

Proof (cf. [31])**.** Let us consider the functions

$$u = \mathcal{A}(\omega)A^{-1}(\omega)(\tfrac{1}{2}I + B(\omega))\psi \quad \text{and} \quad v = \mathcal{B}(\omega)\psi,$$

where $\psi \in H^1(S)$. According to (1.33^+) and (1.37^+), they are solutions of the same problem $D^+(\omega)$, and, due to the assumption of the Proposition, coincide. Thus $Tu^+ = Tv^+$. With the account of (1.34^+) we get

$$(-\tfrac{1}{2}I + B'(\omega))A^{-1}(\omega)(\tfrac{1}{2}I + B(\omega))\psi = T\mathcal{B}(\omega)\psi^+. \tag{1.105}$$

Likewise, the functions

$$u = \mathcal{A}(\omega)A^{-1}(\omega)(-\tfrac{1}{2}I + B(\omega))\psi \quad \text{and} \quad v = \mathcal{B}(\omega)\psi$$

are solutions of the same problem $D^-(\omega)$ by (1.33^-) and (1.37^-). Thus, $Tu^- = Tv^-$, and from (1.34^-) we obtain

$$(\tfrac{1}{2}I + B'(\omega))A^{-1}(\omega)(-\tfrac{1}{2}I + B(\omega))\psi = T\mathcal{B}(\omega)\psi^-. \tag{1.106}$$

It remains to apply (1.102)–(1.103).

The first equality in (1.104) can be extended to ω exceptional with respect to the problem D^+ by passing to the limit as $\omega'' \to 0$.

Corollary 1.10. *For the operator* (1.61) *we have*

$$T^{-1}(\omega)\psi = -T\mathcal{B}(\omega)\psi^+ = -T\mathcal{B}(\omega)\psi^-. \tag{1.107}$$

Generalizations to Lipschitz domains will be indicated in Subsections 2.5 and 2.6.

1.8. *Spectral Properties of Operator $A(\omega)$*

We start with simple remarks on operators in a Hilbert space $\mathcal{H}$.

For any bounded operator $\mathcal{T}$ in $\mathcal{H}$ we have

$$\mathcal{T}^* = (\overline{\mathcal{T}})' = \overline{\mathcal{T}'}, \tag{1.108}$$

where $\overline{\mathcal{T}}v = \overline{\mathcal{T}\overline{v}}$ is the complex conjugate of $\mathcal{T}$ and $\mathcal{T}'$ is the transpose of $\mathcal{T}$. If $\mathcal{T}$ is an integral operator, then $\overline{\mathcal{T}}$ is obtained by replacing its kernel by the complex conjugate kernel.

The real and imaginary parts of a nonselfadjoint bounded operator $\mathcal{T}$ in $\mathcal{H}$ are defined as

$$\operatorname{Re}\mathcal{T} = \frac{1}{2}(\mathcal{T} + \mathcal{T}^*) \quad \text{and} \quad \operatorname{Im}\mathcal{T} = \frac{1}{2i}(\mathcal{T} - \mathcal{T}^*), \tag{1.109}$$

where $\mathcal{T}^*$ is the adjoint of $\mathcal{T}$. An operator $\mathcal{T}$ is called *dissipative* if $\operatorname{Im}\mathcal{T} \geqslant 0$, i.e., $\operatorname{Im}(\mathcal{T}\varphi, \varphi) \geqslant 0$. Such an operator does not have associated vectors corresponding to real eigenvalues (see [14, Chapter V, §1]).

Theorem 1.11. 1. *For all values of ω the operator $A(\omega)$ is* (*complex*) *symmetric*:

$$A(\omega)' = A(\omega), \qquad A(\omega)^* = \overline{A(\omega)}. \tag{1.110}$$

It is selfadjoint if and only if $\omega' = 0$.

2. $\operatorname{Im} A(\omega)$ *is an operator of order $-\infty$ for real nonzero ω and of order not greater than -3 for nonreal not pure imaginary ω.*

3. *For $\omega' < 0$ the operator $A(\omega)$ is dissipative*; *more precisely,*

$$\operatorname{Im}(A(\omega)\varphi, \varphi)_S > 0 \quad \textit{if} \;\; A(\omega)\varphi \neq 0. \tag{1.111}$$

For $\omega' > 0$ the same is true of the operator $-A(\omega)$.

Proof. 1. The kernel $\mathcal{E}(x - y, \omega)$ of $A(\omega)$ is a symmetric matrix depending on $|x - y|$ alone. Thus the operator conjugate to $A(\omega)$ is obtained from $A(\omega)$ by taking the complex conjugate kernel, i.e., by replacing ω by $\overline{\omega}$ and ik_j by $-i\overline{k_j}$ in the exponents. When $\omega' = 0$, the kernel is real and symmetric. When $\omega' \neq 0$, it is only symmetric.

2. For real ω, the kernel of the operator $\operatorname{Im} A(\omega)$ is obtained by replacing $e^{ik_j|x-y|}/|x-y|$ in the kernel of $A(\omega)$ by the infinitely smooth functions $\sin(k_j|x-y|)/|x-y|$. If ω is neither real nor pure imaginary, the entries of the kernel of $\operatorname{Im} A(\omega)$ have the expansions

$$\mathrm{const}\, |x-y| + \mathrm{const}\, \partial_j \partial_k |x-y|^3 + \dots$$

(since k_j^2/ω^2 are real), and it follows (see [28] or [2, Section 1.6]) that this operator is of order not greater than -3.

3. Let $u = \mathcal{A}(\omega)\varphi$. We use formulas (1.42) with $v = \overline{u}$ and (1.79). Adding them, we obtain

$$-\rho\omega^2 \int_{E_R} |u|^2\, dx = -\int_{E_R} E(u, \overline{u})\, dx - \int_S \varphi \cdot \overline{A(\omega)\varphi}\, dS + \int_{S_R} Tu \cdot \overline{u}\, dS. \tag{1.112}$$

If ω is not real, then, by virtue of the radiation conditions, the integral over S_R tends to zero as $R \to \infty$. Separating the imaginary parts of (1.112) and taking the limit, we obtain

$$\operatorname{Im}(A(\omega)\varphi, \varphi)_S = -2\rho\omega'\omega'' \int_{\mathbb{R}^3} |u|^2\, dx. \tag{1.113}$$

This gives (1.111) if $\omega'\omega'' < 0$ and the similar inequality for $-A(\omega)$ if $\omega'\omega'' > 0$. Also, if the left-hand side of (1.113) vanishes, then $u = 0$ everywhere, and hence $\varphi = Tu^- - Tu^+ = 0$.

Now let ω be real and nonzero. Using (1.112) and the corollaries of the radiation conditions, we get

$$\operatorname{Im}(A(\omega)\varphi, \varphi)_S = -\int_{S_R} (k_1(\lambda + 2\mu)|u^{(p)}|^2 + k_2\mu |u^{(s)}|^2)\, dS + O(R^{-1}). \tag{1.114}$$

The right-hand side has a finite limit as $R \to \infty$, since the left-hand side is independent of R. If the limit differs from zero, then its sign is opposite to the sign of ω. If the limit is zero, then

$$\int_{S_R} |u^{(p)}|^2 ds \to 0 \quad \text{and} \quad \int_{S_R} |u^{(s)}|^2 ds \to 0, \tag{1.115}$$

and by the well-known theorem for solutions of the Helmholtz equation (e.g., see [8, §3.3]) we conclude that $u^{(p)}$ and $u^{(s)}$ vanish in G^-. Then $u^- = A(\omega)\varphi = 0$.

Let us also note that due to (1.111) and (1.84), for $\omega' \neq 0$ the operator $A(\omega)$ has the (unique) real eigenvalue at 0 if and only if ω is exceptional with respect to the problem D^+. Since $A(\omega)$ or $-A(\omega)$ is dissipative, the corresponding root space contains only eigenfunctions and is, as we have seen, of finite dimension.

If ω is not exceptional with respect to the problem D^+, then $A(\omega)$ has an unbounded inverse of order 1. If ω is exceptional with respect to the problem D^+, then $\operatorname{Ker} A(\omega)$ can be eliminated by adding to $A(\omega)$ a finite-dimensional operator without perturbing the root functions and other eigenvalues. Such a perturbation does not change the asymptotics of the eigenvalues and plays no role in the questions as to whether the root functions are complete or have some properties of a basis. The operator $A^{-1}(\omega)$ can be made selfadjoint if $\omega' = 0$. If $\omega' \neq 0$, we can assume that it differs from a selfadjoint operator (namely, from its real part) by an operator of order $-\infty$ if ω is real, and by an operator of order -1 if ω is not real.

Together with the ellipticity established in Subsection 1.6, this leads to the following results. Let us number the eigenvalues of $A(\omega)$, with the account of multiplicities, starting from the zero eigenvalue (if it exists) and further in the nonincreasing order of moduli of nonzero eigenvalues. The characteristic numbers, i.e., the inverses of nonzero eigenvalues $\zeta_j = \zeta_j(A(\omega))$, will be denoted by $z_j = z_j(A(\omega))$. Since the principal symbol of $A(\omega)$ is a negative definite matrix (see (1.83)), the characteristic numbers tend to $-\infty$.

Theorem 1.12. *For all ω,*

$$z_j(A(\omega)) = -c(A)j^{1/2} + O(1) \quad as \ j \to \infty \tag{1.116}$$

with a positive constant $c(A)$ independent of ω. In addition, if $\omega' \neq 0$,

$$|\operatorname{Im} z_j| = O(j^{-h}) \quad as \ j \to \infty, \tag{1.117}$$

where h is an arbitrarily large positive number for real ω, and $h = 1/2$ for nonreal ω. The signs of $\operatorname{Im} z_j$ coincide with the sign of ω'. If $|\omega'| < \omega''$, then the $\operatorname{Re} z_j$ are negative.

Proof. In the case of a selfadjoint operator $A(\omega)$, formula (1.116) follows from the results of [17]. Moreover, the eigenvalues of the principal symbol have constant multiplicities; thus it admits a smooth diagonalization, and the result essentially follows from [16]. If $A(\omega)$ is nonselfadjoint, then the asymptotics (1.116) is first obtained for the characteristic numbers of its real part, and then for the characteristic numbers of $A(\omega)$ using a theorem from [24]. The estimate (1.117) is obtained very easily: we use the invertibility of operators sufficiently close in norm to an invertible one, see [1]. The statement about the signs of $\operatorname{Im} z_j$ follows from Statement 3 of Theorem 1.11. The fact that $\operatorname{Re} z_j$ are negative for $|\omega'| \leqslant \omega''$ follows from the formula

$$\operatorname{Re} \int_S A(\omega)\varphi \cdot \overline{\varphi}\, dS = -\int_{\mathbb{R}^3} E(u, \overline{u})\, dx + \rho((\omega')^2 - (\omega'')^2) \int_{\mathbb{R}^3} |u|^2\, dx, \tag{1.118}$$

which in turn follows from (1.112).

If $\omega' = 0$, then $A(\omega)$ is selfadjoint, and its eigenfunctions form an orthonormal basis $\{g_j\}_{j=1}^{\infty}$ in $L_2(S)$. This basis remains an unconditional basis in all spaces $H^t(S)$. Moreover, we can introduce an inner product in $H^t(S)$ in which this basis remains orthogonal. In the Fourier series of an arbitrary function g,

$$g = \sum_{j=1}^{\infty} c_j g_j, \qquad c_j = (g, g_j)_S, \tag{1.119}$$

the rate of decay of the Fourier coefficients grows with the smoothness of g, and for an infinitely smooth g they decay in absolute value faster than any negative power of j.

Theorem 1.13. *If $\omega' \neq 0$, then the root functions of the operator $A(\omega)$ are complete in all $H^t(S)$. Moreover, there exists a system $\{g_j\}_{j=1}^{\infty}$ of root functions that is an unconditional basis with parentheses simultaneously in all $H^t(S)$.*

We refer the reader to [2] and references therein for terminology and theorems on nonselfadjoint operators. We only recall that the completeness of a set of vectors in a topological space means that their finite linear combinations are dense. Note also that, instead of (1.119), now we have

$$g = \sum_{l=1}^{\infty} \Bigg[\sum_{j=j_l+1}^{j_{l+1}} c_j g_j \Bigg], \qquad c_j = (g, h_j)_S, \tag{1.120}$$

where the g_j belong to the root space corresponding to the eigenvalue ζ_j. Here $\{h_k\}$ is the sequence of functions biorthogonal to $\{g_j\}$, which is composed of the root functions of the operator $\overline{A(\omega)}$ (adjoint to $A(\omega)$), and $\{j_l\}$ is a certain increasing sequence of positive integers independent of g. The square brackets in (1.120) single out the terms corresponding to close characteristic numbers. The series in l converges unconditionally, i.e., it admits any permutation of terms.

Remark 1.14. The statement of Theorem 1.13 can be supplemented with the information on the control over the positions of the brackets in (1.120) and the rate of convergence of the series in l. We refer the interested reader to [2]. Note only that in the case of a real ω, when the operator is infinitely close to a selfadjoint one, one can consider the difference of the series (1.120) and a similar series with parentheses in the eigenfunctions of $\operatorname{Re} A(\omega)$, and this difference converges rapidly in $H^t(S)$ for $g \in H^t(S)$. In the abstract theorems for unbounded operators $\mathcal{T}$, from which these facts follow, the essential role is played by the quantity $p(1-q)$, where p is the exponent in the eigenvalue asymptotics of $\operatorname{Re}\mathcal{T}$ and q is the order of subordination of $\mathcal{T} - \operatorname{Re}\mathcal{T}$ to $\operatorname{Re}\mathcal{T}$. In our case $\mathcal{T} = A^{-1}(\omega)$, $p = 1/2$, and $q = -1$ or $-\infty$, so that $p(1-q) = 1$ (the good case in which ω is neither real nor pure imaginary) or $p(1-q) = -\infty$ (the very good case of real nonzero ω).

Similar remarks pertain to Theorems 1.18 and 1.22 below.

1.9. *Spectral Properties of Operators $T^{\pm}(\omega)$*

In this subsection we assume that ω is not exceptional with respect to the problem N^+ when considering the operator $T^+(\omega)$ and with respect to the problem D^+ when considering the operator $T^-(\omega)$.

Theorem 1.15. 1. *The operators $T^{\pm}(\omega)$ are symmetric for all values of ω:*

$$(T^{\pm})'(\omega) = T^{\pm}(\omega), \qquad (T^{\pm})^*(\omega) = \overline{T^{\pm}(\omega)}. \tag{1.121}$$

The operator $T^+(\omega)$ is selfadjoint if and only if $\omega'\omega'' = 0$; the operator $T^-(\omega)$ is selfadjoint if and only if $\omega' = 0$.

2. *For real nonzero ω, the imaginary part of $T^-(\omega)$ is of order $-\infty$. For $\omega'\omega'' \neq 0$, the imaginary parts of $T^+(\omega)$ and $T^-(\omega)$ are of order not greater than -3.*

3. *For $\omega' < 0$ the operator $T^-(\omega)$ is dissipative, and moreover,*

$$\operatorname{Im}(T^-(\omega)\varphi, \varphi)_S > 0 \quad \text{if } \varphi \neq 0. \tag{1.122}$$

For $\omega' > 0$ the same is true of the operator $-T^-(\omega)$. Similar statements hold for $T^+(\omega)$ with nonreal ω.

Proof. 1. By (1.42) with $v = \overline{u}$ and (1.48) with the account of

$$T\overline{u}^+ = \overline{Tu^+}, \tag{1.123}$$

separating the imaginary parts, we obtain

$$\operatorname{Im}(T^+(\omega)\varphi, \varphi) = -2\omega'\omega''\rho \int_{G^+} |u|^2 dx, \tag{1.124}$$

where $\varphi = Tu^+$ is arbitrary. We see that $T^+(\omega)$ is selfadjoint if and only if $\omega'\omega'' = 0$. For $T^-(\omega)$ with $\omega \neq 0$ we use (1.79). If $\omega'' > 0$, we obtain

$$\operatorname{Im}(T^-(\omega)\varphi, \varphi)_S = -2\omega'\omega''\rho \int_{G^-} |u|^2 dx \tag{1.125}$$

with arbitrary $\varphi = Tu^-$ and conclude that $T^-(\omega)$ is selfadjoint if and only if $\omega' = 0$. If $\omega'' = 0$, we obtain

$$\operatorname{Im}(T^-(\omega)\varphi, \varphi)_S = -\int_{S_R} (k_1(\lambda + 2\mu)|u^{(p)}|^2 + k_2\mu|u^{(s)}|^2)\, dS + O(R^{-1}) \tag{1.126}$$

(cf. (1.114)) and conclude that $T^-(\omega)$ is nonselfadjoint. Furthermore, by (1.102) and (1.103) with the account of (1.110), we obtain

$$((\tfrac{1}{2}I \pm B(\omega))^{-1}A(\omega))' = A(\omega)(\tfrac{1}{2}I \pm B'(\omega))^{-1} = (\tfrac{1}{2}I \pm B(\omega))^{-1}A(\omega).$$

In particular, we see that $T^-(0)$ is selfadjoint.

2. By the second equality in (1.121),

$$\operatorname{Im} T^{\pm}(\omega) = \tfrac{1}{2i}(T^{\pm}(\omega) - \overline{T^{\pm}(\omega)}). \tag{1.127}$$

Here

$$T^-(\omega) - \overline{T^-(\omega)} = (\tfrac{1}{2}I + B(\omega))^{-1}(A(\omega) - \overline{A(\omega)}) + ((\tfrac{1}{2}I + B(\omega))^{-1} - (\tfrac{1}{2}I + \overline{B(\omega)})^{-1})\overline{A(\omega)}, \tag{1.128}$$

and

$$(\tfrac{1}{2}I + B(\omega))^{-1} - (\tfrac{1}{2}I + \overline{B(\omega)})^{-1} = (\tfrac{1}{2}I + B(\omega))^{-1}(\overline{B(\omega)} - B(\omega))(\tfrac{1}{2}I + \overline{B(\omega)})^{-1}. \tag{1.129}$$

It is easy to check that the operator $\overline{B(\omega)} - B(\omega)$ is of order $-\infty$ for a real nonzero ω, and of order not greater than -2 for $\omega'\omega'' \neq 0$. Thus, (1.127)–(1.129) imply our statement for $T^-(\omega)$. The case of $T^+(\omega)$ is treated similarly.

3. These statements follow from (1.124)–(1.126).

Proposition 1.16. *The principal symbols of the operators $T^{\pm}(\omega)$ are Hermitian negative definite matrices. In particular, $T^{\pm}(\omega)$ are elliptic operators of order -1 with index zero.*

Proof. We denote by $a(\xi')$, $b(\xi')$, and $c(\xi')$ the principal symbols of the operators $A(\omega)$, $B(\omega)$, and $\frac{1}{2}I \pm B(\omega)$, respectively, at a point on S (see Subsection 1.6). These matrices are Hermitian, $a(\xi')$ is negative definite, and $c(\xi')$ is positive definite. The principal symbol of $B'(\omega)$ is $b'(-\xi')$, it coincides with $b(\xi')$, and it follows from (1.99) that $a(\xi')$ and $b(\xi')$ commute (which, of course, can also be checked directly). Hence $c^{-1}(\xi')$ and $a(\xi')$ also commute, and $c^{-1}(\xi')a(\xi')$ is a Hermitian matrix. For any vector $v = (v_1, v_2)' \neq 0$, we have

$$c^{-1}(\xi')a(\xi')v \cdot \overline{v} = a(\xi')(c^{-1/2}v) \cdot \overline{(c^{-1/2}v)} < 0,$$

and hence $c^{-1}(\xi')a(\xi')$ is negative definite.

We can now see that the spectral properties of the operators $T^{\pm}(\omega)$ are very close to those of the operator $A(\omega)$. In particular, $T^+(\omega)$ has eigenvalue zero for the same ω as $A(\omega)$, and the eigenvalues of $T^-(\omega)$ are always nonzero (due to our assumption of invertibility of $\frac{1}{2}I + B(\omega)$). If 0 is an eigenvalue of $A(\omega)$, then $\operatorname{Ker} T^+(\omega) = \operatorname{Ker} A(\omega)$. No other real eigenvalues are possible if $T^+(\omega)$ or $T^-(\omega)$ is nonselfadjoint.

Let us number the eigenvalues as in Subsection 1.8. Let $z_j(T^{\pm}(\omega))$ be the characteristic numbers of $T^{\pm}(\omega)$. We have the following results.

Theorem 1.17. *For all ω,*

$$z_j(T^{\pm}(\omega)) = -c(T^{\pm})j^{1/2} + O(1) \quad \text{as } j \to \infty \tag{1.130}$$

with positive constants $c(T^{\pm})$ independent of ω. In addition, if the operators $T^{\pm}(\omega)$ are nonselfadjoint, then

$$|\operatorname{Im} z_j| = O(j^{-h}) \quad \text{as } j \to \infty, \tag{1.131}$$

where $h = 1/2$ for $\omega'\omega'' \neq 0$, and h is an arbitrarily large positive number for real nonzero ω in the case of the operator $T^-(\omega)$. If $\omega'\omega'' \neq 0$ for $T^+(\omega)$ or $\omega' \neq 0$ for $T^-(\omega)$, then the signs of $\operatorname{Im} z_j$ coincide with the sign of ω'. If $|\omega'| < \omega''$, then $\operatorname{Re} z_j$ are all negative.

Theorem 1.18. *If $\omega'\omega'' \neq 0$, then the root functions of the operator $T^+(\omega)$ are complete in all $H^t(S)$. Moreover, there exists a system $\{g_j\}_{j=1}^{\infty}$ of root functions that is an unconditional basis with parentheses simultaneously in all $H^t(S)$. The same is true of $T^-(\omega)$ if $\omega' \neq 0$.*

1.10. *Spectral Properties of Operator* $T(\omega)$

Here we assume that ω is not exceptional with respect to both interior problems. The spectral properties of the operator $T(\omega)$ are similar to those of $A(\omega)$ and $T^-(\omega)$. Theorem 1.19 and Proposition 1.20 are obtained below as direct applications of the results of Subsection 1.9 and formula (1.61). Theorems 1.21 and 1.22 follow as corollaries.

Theorem 1.19. 1. *The operator* $T(\omega)$ *is symmetric*:

$$T'(\omega) = T(\omega), \qquad T^*(\omega) = \overline{T(\omega)}. \tag{1.132}$$

It is selfadjoint if and only if $\omega' = 0$.

2. *The imaginary part of* $T(\omega)$ *is of order* $-\infty$ *for real nonzero* ω *and of order not greater than* -3 *for nonreal and not pure imaginary* ω.

3. *For* $\omega' < 0$ *the operator* $T(\omega)$ *is dissipative, and moreover,*

$$\operatorname{Im}(T(\omega)\varphi, \varphi)_S > 0 \quad \textit{if } \varphi \neq 0. \tag{1.133}$$

For $\omega' > 0$ *the same is true of the operator* $-T(\omega)$.

Proposition 1.20. *The principal symbol of* $T(\omega)$ *is a Hermitian negative definite matrix.*

We number the characteristic numbers of $T(\omega)$ as above.

Theorem 1.21. *For all admissible* ω, *the characteristic numbers of* $T(\omega)$ *have the asymptotics*

$$z_j(T(\omega)) = -c(T)j^{1/2} + O(1) \quad \textit{as } j \to \infty \tag{1.134}$$

with a positive constant $c(T)$ *independent of* ω. *In addition, if* $\omega' \neq 0$, *then*

$$|\operatorname{Im} z_j| = O(j^{-h}) \quad \textit{as } j \to \infty, \tag{1.135}$$

where h *is an arbitrarily large positive number for real* ω, *and* $h = 1/2$ *for nonreal* ω. *If* $\omega' \neq 0$, *then the signs of* $\operatorname{Im} z_j(T(\omega))$ *coincide with the sign of* ω'. *If* $|\omega'| < \omega''$, *then* $\operatorname{Re} z_j(T(\omega)) < 0$ *for all* j.

Theorem 1.22. *If* $\omega' \neq 0$, *then the root functions of the operator* $T(\omega)$ *are complete in all* $H^t(S)$. *Moreover, there exists a system* $\{g_j\}_{j=1}^{\infty}$ *of root functions that is an unconditional basis with parentheses simultaneously in all* $H^t(S)$.

1.11. *Problems* I–IV *for Eceptional* ω

Problem III did not cause any difficulties from the very beginning; it has already been considered for all ω with $\operatorname{Im}\omega \geqslant 0$.

Let us consider Problem I. So far, we assumed ω to be nonexceptional with respect to the problem N^+. If this condition is violated but ω is nonexceptional with respect to the problem D^+, then Problem I can be reduced to the equation

$$\Theta^+(\omega)\varphi = z\varphi, \tag{1.136}$$

where

$$\Theta^+(\omega) = A^{-1}(\omega)(\tfrac{1}{2}I - B(\omega)), \quad \varphi = u^+, \quad z = \zeta^{-1}. \tag{1.137}$$

This operator is an elliptic ΨDO of order 1 with compact resolvent, selfadjoint for $\omega'\omega'' = 0$. It is easy to check (cf. [1, 2]) that it differs from its real part by an operator of order -1 for the other ω in the upper half-plane. The spectral results are similar to those obtained in Subsection 1.9. The operator $\Theta^+(\omega)$ has the eigenvalue zero, and therefore the "multi-valued inverse operator" $T^+(\omega)$ has the "eigenvalue" ∞.

However, one cannot exclude the case of the "double resonance," when for a given ω both problems $D^+(\omega)$ and $N^+(\omega)$ have nontrivial solutions.

In this case we can propose the following modification of the problem. Let us replace ζ by $1/z$ in (1.6), multiply this boundary condition by z, replace z by $\tilde{z}+h$, set $\tilde{\zeta} = 1/\tilde{z}$, multiply the result by $\tilde{\zeta}$, and, finally, replace $\tilde{\zeta}$ by ζ. We arrive at the boundary condition

$$u^+ + \zeta(Tu^+ + hu^+) = 0. \tag{1.138}$$

The role of the problem $N^+(\omega)$ is now played by the interior problem with the boundary condition

$$Tu^+ + hu^+ = g, \tag{1.139}$$

and it is sufficient to choose a number h in such a way that this boundary value problem will be uniquely solvable. Since we have added only a lower-order term to the boundary condition, the index of problem (1.3), (1.139) is still zero, and we should only take care of the uniqueness. Formula (1.42) with $v = \overline{u}$ shows that it suffices to take a nonreal h. Formula (1.46^+) shows that the modified Problem I is reduced to the equation

$$-\zeta(B(\omega) + hA(\omega) - \tfrac{1}{2}I)\varphi = A(\omega)\varphi, \tag{1.140}$$

where $\varphi = Tu^+ + hu^+$. The operator on the left-hand side still has index zero, and its null space is trivial for nonreal h. (Indeed, if φ belongs to this null space, then we consider $u = \mathcal{B}(\omega)\varphi + h\mathcal{A}(\omega)\varphi$; for this function we have $u^- = 0$, $u \equiv 0$ in G^-, $Tu^- = 0$, $u^+ = \varphi$, and $Tu^+ = h\varphi$ (by the formulas of Subsection 1.2). Now, separating the imaginary parts is Green's formula (1.42) with $\overline{v} = u$, we see that $\varphi = 0$.) Multiplying (1.140) by the inverse operator, we finally arrive at

$$T^+(\omega, h)\varphi = \zeta\varphi, \tag{1.141}$$

where

$$T^+(\omega, h) = (\tfrac{1}{2}I - B(\omega) - hA(\omega))^{-1}A(\omega). \tag{1.142}$$

However, it is not necessary to take h pure imaginary. If h_1 is not a characteristic number of the compact operator $T^+(\omega, i)$, then the operator $\frac{1}{2}I - B(\omega) - (i + h_1)A$ is also invertible, so we can find a suitable real h. We omit the analysis of the resulting operator.

Now let us consider Problem II. Let ω be exceptional with respect to the problem D^+. The operator $T^-(\omega)$ (see (1.52)) exists as before, but now we cannot obtain the representation (1.51) for it. We derive a different representation. Without loss of generality we assume that the domain G^+ contains the origin and denote by G_r^+ a ball of small radius r centered at the origin and contained (together with its boundary S_r) in G^+. For any given ω and a sufficiently small r, ω is not exceptional with respect to the problem D^+ in G_r^+. Indeed, if $u(x)$ is a solution of the system (1.3) in this ball, then $v(x) = u(rx)$ is the solution of the system $L(\partial_x)v(x) + \rho(r\omega)^2v(x) = 0$ in the unit ball $\{x : |x| < 1\}$, and it suffices to assume that $\rho(r\omega)^2$ is less than the first eigenvalue of $-L_D$ in this ball.

Now consider the problem

$$L(\partial_x, \omega)u + \rho\omega^2 u = F \quad \text{in } G_r^-, \qquad \beta Tu^- - u^- = 0 \quad \text{on } S_r, \tag{1.143}$$

where G_r^- is the complement of $\overline{G_r^+}$, F is compactly supported, $T = T(\partial_x, \nu(x))$, where $\nu(x)$ in the exterior normal unit vector to S_r, and the radiation conditions are imposed at infinity. For definiteness, let $\omega > 0$. Then for $\operatorname{Im}\beta > 0$ this problem can be solved uniquely (see Theorem 1.17). Let $\mathcal{G}(x, y, \omega)$ be Green's function of this problem. It differs from $\mathcal{E}(x - y, \omega)$ by a term $g(x, y, \omega)$ which is infinitely smooth for $x, y \notin S_r$.

Let us define the potentials $\mathcal{A}_r(\omega)$ and $\mathcal{B}_r(\omega)$ and the operators $A_r(\omega)$ and $B_r(\omega)$ by formulas of the form (1.29)–(1.32) with kernel $\mathcal{G}(x, y, \omega)$ instead of $\mathcal{E}(x - y, \omega)$. Instead of (1.46$^-$) we now have

$$(B_r(\omega) + \tfrac{1}{2}I)u^- = A_r(\omega)Tu^-. \tag{1.144}$$

The analog of the homogeneous problem $D^+(\omega)$ now is

$$\begin{gathered} L(\partial_x, \omega)u = 0 \quad \text{in } G^+ \setminus \overline{G_r^+}, \\ u^+ = 0 \quad \text{on } S, \qquad \beta Tu^- - u^- = 0 \quad \text{on } S_r, \end{gathered} \tag{1.145}$$

and this problem does not have nontrivial solutions, which is checked with the use of Green's formula. Hence the operator on the left-hand side in (1.144) is invertible (cf. Subsection 1.6), and for $T^-(\omega)$ we obtain the representation

$$T^-(\omega) = (\tfrac{1}{2}I + B_r(\omega))^{-1}A_r(\omega) \tag{1.146}$$

which allows us to obtain the same results as for the values of ω not exceptional with respect to the problem D^+.

Concerning Problem IV, we only make the following remark. The operator (1.104) can be considered under the condition of invertibility of $A(\omega)$, i.e., under the only assumption that ω is not exceptional with respect to the problem D^+. Under this assumption, we can reduce Problem IV to the equation

$$\Theta(\omega)\varphi = z\varphi, \tag{1.147}$$

where $\varphi = u^- - u^+$ and

$$\Theta(\omega) = A^{-1}(\omega)(\tfrac{1}{4}I - B^2(\omega)). \tag{1.148}$$

This is an elliptic operator of order 1 which has the inverse (1.61) if ω is not exceptional with respect to the problem N^+.

2. DOMAINS WITH LIPSCHITZ BOUNDARY

2.1. *Coordinate Cylinders. Nontangential Convergence. Function Spaces*

Let us recall that a surface S is called *Lipschitz* if in some neighborhood $Z(x_0)$ of every point $x_0 \in S$ it can be identified (after a suitable rotation of the coordinate system in $\mathbb{R}^3$) with the graph of a function $x_3 = X(x') = X(x_1, x_2)$ satisfying the Lipschitz condition

$$|X(x') - X(y')| \leqslant C|x' - y'|, \tag{2.1}$$

where C is a Lipschitz constant independent of x', y'. It is convenient to assume that the neighborhood $Z(x_0)$ is a cylinder

$$Z = Z(x_0) = \{x \in \mathbb{R}^3 : |x' - x_0'| < r, \ |x_3 - x_{0,3}| < s\} \tag{2.2}$$

and that (2.1) holds for $|x' - x_0'| < r$. Assume additionally that $2Cr < s$; then, in particular, the graph intersects only the lateral surface of the cylinder. Assume also that the parts of the neighborhood (2.2) lying below and above the graph of the function X are subsets of $G^\pm$, respectively. We include all the conditions above in the notion of a *coordinate cylindrical neighborhood* $Z(x_0)$. A finite union of such cylinders with fixed r and s covers S. Let $\varkappa Z(x_0)$ denote the image of the neighborhood (2.2) under the dilation with center x_0 and coefficient $\varkappa$. It is convenient to assume that $2Z(x_0)$ remains a coordinate cylindrical neighborhood of x_0.

The functions satisfying the Lipschitz condition are differentiable almost everywhere (e.g., see [30, Chapter 8]) and have bounded gradients; therefore the Lebesgue measure in $\mathbb{R}^3$ induces a Lebesgue measure on S with the area element $dS_x = (1 + |\nabla X(x')|^2)^{1/2} dx'$ in the local coordinates. The unit exterior normal vector $\nu(x)$ is defined almost everywhere on S; we can assume that for $x \in Z(x_0) \cap S$ it forms an acute angle with the axis of $Z(x_0)$, which is not greater than a fixed angle $\alpha < \pi/2$.

Let us introduce the sets

$$\Gamma^{\pm}(x) = \{y \in G^{\pm} : |x - y| < \beta r(y, S)\} \tag{2.3}$$

for points $x \in S$, where $r(y, S)$ is the distance from y to S and β is a sufficiently large fixed number. We can assume that for all $x \in S \cap Z(x_0)$ two finite circular cones of fixed size with vertex at x and axes parallel to the axis of $Z(x_0)$ are contained in $\Gamma^{\pm}(x)$.

The boundary conditions of Problems I–IV preserve their sense (almost everywhere) if we now understand $u^+(x)$ and $u^-(x)$ as the limit values of $u(y)$ as $y \to x$ in $\Gamma^+(x)$ or $\Gamma^-(x)$, respectively, for almost all $x \in S$. Such convergence is called *nontangential*. The derivatives $\partial_\nu u^+(x)$ and $\partial_\nu u^-(x)$ are understood as the limits $\partial u(y)/\partial\nu(x)$ as $y \to x$ in $\Gamma^{\pm}(x)$ for almost all $x \in S$. Here $\nu(x)$ can be replaced by $\nu(y)$ with $y = \Lambda_j^{\pm}(x)$ being the image of the point $x \in S$ under a Lipschitz diffeomorphism $\Lambda_j^{\pm} : S \to S_j^{\pm}$ of the surface S onto an approximating closed infinitely smooth surface $S_j^{\pm} \subset G^{\pm}$ and $\nu(y) = \nu_j^{\pm}(y)$ being the unit exterior normal vector to $S_j^{\pm}$ at y (here we take into account a lemma, e.g., from [33, p. 581]).

As we have already noted, the Sobolev spaces $H^t(S)$ are now intrinsically defined only for $|t| \leqslant 1$. We shall use only the spaces $H^0(S)$ and $H^1(S)$ and the spaces $H^t(G^+)$ and $H^t_{\mathrm{loc}}(G^-)$ for $0 \leqslant t \leqslant 2$. Note that the trace operator on the Lipschitz surface S is a bounded operator from $H^t(G^+)$ or $H^t_{\mathrm{loc}}(G^-)$ into $H^{t-1/2}(S)$ only for $1/2 < t < 3/2$ in the general case (e.g., see [18, 12]), but it is possible to consider the critical values $t = 1/2$ and $t = 3/2$ for solutions of some elliptic equations including the Lamé system (see below). Also, the trace operator is bounded from $H^t(G^+)$ or $H^t(G^-)$ into $H^1(S)$ for $t > 3/2$ (e.g., see [12]).

For a vector-valued function $w(x)$ defined in G^+ or G^-, we define the *nontangential maximal functions*

$$w_*^{\pm}(x) = \sup_{y \in \Gamma^{\pm}(x)} |w(y)| \qquad (x \in S). \tag{2.4}$$

Here instead of (2.3) a regular family of nontangential cones can also be used (e.g., see [32, 33]).

Let us now introduce four function spaces (cf. [31, 4]).

The space $V^+(\omega)$ consists of all solutions $u(x)$ of system (1.3) in G^+ that are infinitely smooth in G^+ and have nontangential limits almost everywhere on S with $u_*^+ \in L_2(S)$. We can use the L_2-norm $\|u_*^+\|$ as the norm in $V^+(\omega)$, because if it is equal to 0, then $u(x) = 0$ in a boundary strip and, therefore, in all of G^+.

The space $V^-(\omega)$ consists of all solutions of system (1.3) in G^- that are infinitely smooth in G^-, have nontangential limits almost everywhere on S with $u_*^- \in L_2(S)$ and satisfy the radiation conditions. We equip $V^-(\omega)$ with the norm $\|u_*^-\|$.

The space $W^+(\omega)$ consists of all solutions of system (1.3) in G^+ that are infinitely smooth in G^+, have nontangential limits for u and ∇u almost everywhere on S, and have a finite norm $\|u_*^+\| + \|\nabla u_*^+\|$ in $L_2(S)$.

Here and below ∇u is the matrix of first partial derivatives of the components of the vector u, and $\|\nabla u\|^2$ is the sum of the squares of the norms of the entries of this matrix.

Finally, the space $W^-(\omega)$ consists of all solutions of system (1.3) in G^- that are infinitely smooth in G^-, have nontangential limits for u and ∇u almost everywhere on S, have a finite norm $\|u_*^-\| + \|\nabla u_*^-\|$ in $L_2(S)$, and satisfy the radiation conditions.

The spaces of this type are convenient in potential theory for Lipschitz domains. In particular, these spaces (with $\omega = 0$) are essentially used in [10]. For us, their relations to the Sobolev spaces in $G^{\pm}$ are important.

The general results of [11] for homogeneous elliptic systems imply that a solution $u(x)$ of system (1.3) with $\omega = 0$ in a domain G^+ with connected boundary belongs to $V^+(0)$ if and only if

$$\int_{G^+} r(x, S)|\nabla u(x)|^2 dx < \infty. \tag{2.5}$$

We shall need the following theorem, close to some known results (cf. [10, p. 796]).

Theorem 2.1. *A solution $u(x)$ of system* (1.3) *belongs to $V^+(\omega)$ if and only if it belongs to $H^{1/2}(G^+)$.*

Proof. First we check this statement for $\omega = 0$ following [18], where a similar result is derived for the Laplace equation from the results of [9]; then we extend the theorem to all ω in the upper half-plane.

It was proved in [18] without the assumption of harmonicity that condition (2.5) implies the inclusion $u(x) \in H^{1/2}(G^+)$. Thus, in the case $\omega = 0$ it remains only to check for solutions of the Lamé system that $u \in H^{1/2}(G^+)$ implies (2.5). To prove this (see (2.8) below), we use the mean value theorem for the Lamé system for $\omega = 0$ (cf. [22, Chapter 14, §1]). This theorem says that

$$u(x) = \frac{1}{4\pi r^2} \int_{|y-x|=r} (C_1 E + C_2 \Lambda(y-x)) u(y) \, dS_y, \tag{2.6}$$

where C_1 and C_2 are some constants, E is the 3×3 identity matrix, and $\Lambda(y)$ is the matrix with entries $y_j y_k / |y|^2$. Multiplying (2.6) by r^2, integrating along $[0, R]$, and dividing by $R^3/3$, we obtain

$$u(x) = \frac{1}{\frac{4}{3}\pi R^3} \int_{|y-x| \leqslant R} (C_1 E + C_2 \Lambda(y-x)) u(y) \, dy. \tag{2.7}$$

Since the Lamé system has constant coefficients, the derivatives of $u(x)$ also satisfy it, and therefore

$$\begin{aligned} \partial_l u(x) = {} & \frac{1}{\frac{4}{3}\pi R^3} \int_{|y-x| \leqslant R} \partial_{y_l} ((C_1 E + C_2 \Lambda(y-x))(u(y) - u(x))) \, dy \\ & - \frac{1}{\frac{4}{3}\pi R^3} \int_{|y-x| \leqslant R} C_2 \partial_{y_l} (\Lambda(y-x))(u(y) - u(x)) \, dy. \end{aligned}$$

Applying the divergence theorem to the second integral, we obtain

$$|\partial_l u(x)| \leqslant C_3 R^{-3} \biggl(\int_{|y-x|=R} |u(y) - u(x)| \, dS + \int_{|y-x| \leqslant R} \frac{|u(y) - u(x)|}{|y-x|} \, dy \biggr).$$

Here we replace R by r, multiply the equation by r^3, and integrate both sides again from 0 to R, estimating beforehand the triple integral by its maximum value, the integral over the ball of radius R. Dividing by R^4, we obtain

$$\begin{aligned} |\partial_l u(x)| & \leqslant C_4 \biggl(R^{-4} \int_{|y-x| \leqslant R} |u(y) - u(x)| \, dy + R^{-3} \int_{|y-x| \leqslant R} \frac{|u(y) - u(x)|}{|y-x|} \, dy \biggr) \\ & \leqslant C_5 R^{-2} \int_{|y-x| \leqslant R} \frac{|u(y) - u(x)|}{|y-x|^2} \, dy. \end{aligned}$$

Applying the Schwarz inequality, we get

$$R \, |\partial_l u(x)|^2 \leqslant C_6 \int_{|y-x| \leqslant R} \frac{|u(y) - u(x)|^2}{|y-x|^4} \, dy.$$

Let us replace R by $r(x, S)/2$; integrating with respect to x, we obviously obtain the desired inequality

$$\int_{G^+} r(x, S) |\nabla u(x)|^2 \, dx \leqslant C_7 \iint_{G^+ \times G^+} \frac{|u(y) - u(x)|^2}{|y-x|^4} \, dx \, dy. \tag{2.8}$$

Thus, we can now use Theorem 2.1 for $\omega = 0$. Let ω be arbitrary, and let $u(x)$ be a solution of (1.3) from $V^+(\omega)$. Then it belongs to L_2 in a boundary strip and, therefore, in G^+. Consider the volume potential

$$U(x) = \int_{G^+} \mathcal{E}(x-y)F(y)\,dx, \quad \text{where } F = \rho\omega^2 u. \tag{2.9}$$

It is easy to see, using the Schwarz inequality, that U is a bounded function; it is also known that it belongs to $H^2(\mathbb{R}^3)$ and therefore to $H^2(G^+)$. The sum $v = u + U$ solves system (1.3) for $\omega = 0$ and belongs to $V^+(0)$. Thus $v \in H^{1/2}(G^+)$ and, as a result, $u(x) \in H^{1/2}(G^+)$.

Now let $u(x)$ be a solution of (1.3) from $H^{1/2}(G^+)$. Then $v = u+U$ solves system (1.3) for $\omega = 0$ and belongs to $H^{1/2}(G^+)$. Thus $v \in V^+(0)$, and we conclude that $u \in V^+(\omega)$.

The same result holds for the exterior domain:

Corollary 2.2. *$u \in V^-(\omega)$ if and only if u is a solution of* (1.3) *from $H^{1/2}_{\mathrm{loc}}(G^-)$ satisfying the radiation conditions.*

Indeed, it suffices to cover a boundary strip in G^- by a finite union of suitable bounded Lipschitz domains G_j adherent to S with connected boundaries and apply Theorem 2.1 to each of them.

More precisely, the implication $u \in H^{1/2}_{\mathrm{loc}}(G^-) \implies u \in V^-(\omega)$ is obtained without difficulties. Let us describe a possible way of construction of domains G_j to obtain the converse implication $\Longleftarrow$. We fix a coordinate cylinder $Z(x_0)$ and construct a Lipschitz domain $G_0 \subset Z(x_0)$ with connected boundary S_0 consisting of the graphs of two functions, $x_n = X(x')$ (see (2.1)) and $x_n = Y(x')$, where

$$X(x') < Y(x') < X(x') + s/4 \quad \text{for } |x' - x_0'| < r/2$$

and

$$X(x') = Y(x') \quad \text{for } |x' - x_0'| \geqslant r/2.$$

Here $Y(x') = X(x') + z(x')$, where $z(x')$ is, say, an appropriate C^∞ function. Instead of $\Gamma^-(x)$ (see (2.3)), we will temporarily write $\Gamma^-_\beta(x, S)$. Similarly, we set

$$\Gamma^+_\gamma(x, S_0) = \{y \in G_0 : |x-y| < \gamma r(y, S_0)\}.$$

We intend to verify that (for sufficiently large γ) there exist $\beta > \gamma$ independent of x', $|x'-x_0'| < r/2$, such that

$$\Gamma^+_\gamma((x', Y(x')), S_0) \subset \Gamma^-_\beta((x', X(x')), S). \tag{2.10}$$

¿From this we can conclude that if the function $u(x)$ belongs to $V^-(\omega)$ in G^-, then it belongs to $V^+(\omega)$ in G_0 and, by Theorem 2.1, to $H^{1/2}(G_0)$.

To obtain the inclusion (2.10), we consider the set

$$K_\gamma(x, S) = \{y \in G_0 : |y' - x'| \leqslant \gamma r(y, S)\}.$$

For sufficiently large β,

$$K_\gamma(x', S) \subset \Gamma^-_\beta((x', X(x')), S).$$

Indeed, let $\varepsilon \in (0,1)$ be so small (and γ so large) that the cone

$$L_\varepsilon(x) = \{y \in G^- : |y' - x'| < \varepsilon|y - x|,\ x_n < y_n < x_n + s/4\}$$

lies in $\Gamma^-_\gamma(x, S)$ for all x, $|x' - x_0'| < r/2$. Set $\beta = \gamma/\varepsilon$. Of course, the cone is contained in $\Gamma^-_\beta(x, S)$. If $y \in K_\gamma(x', S)$, then either $y \in L_\varepsilon(x) \subset \Gamma^-_\beta(x, S)$ or $y \notin L_\varepsilon(x)$; in the latter case

$$\frac{|y-x|}{\beta} \leqslant \frac{|y'-x'|}{\beta\varepsilon} = \frac{|y'-x'|}{\gamma} \leqslant r(y, S),$$

so that again $y \in \Gamma_\beta^-(x, S)$.

It remains to verify that

$$\Gamma_\gamma((x', Y(x')), S_0) \subset K_\gamma(x', S). \tag{2.11}$$

Assume that y belongs to the left-hand side of (2.11). Then

$$r(y, S) \geqslant r(y, S_0) \geqslant \frac{|y-x|}{\gamma} \geqslant \frac{|y'-x'|}{\gamma}$$

which yields $y \in K_\gamma(x', S)$. $\square$

Corollary 2.3. *A solution $u(x)$ of (1.3) in G^+ belongs to $W^+(\omega)$ if and only if it belongs to $H^{3/2}(G^+)$. Similarly, a solution $u(x)$ of (1.3) in G^- satisfying the radiation conditions belongs to $W^-(\omega)$ if and only if it belongs to $H^{3/2}_{\mathrm{loc}}(G^-)$.*

This is obtained by applying Theorem 2.1 and Corollary 2.2 to the first partial derivatives of $u(x)$.

2.2. *Integral Operators*

Proposition 2.4. *For $\omega = 0$:*

1. *The operators B and B' are bounded in $H^0(S)$, B is also bounded in $H^1(S)$, and A is bounded as an operator from $H^0(S)$ into $H^1(S)$.*
2. *The operator $\mathcal{B}$ is bounded from $H^0(S)$ into $V^\pm(0)$ and from $H^1(S)$ into $W^\pm(0)$, and $\mathcal{A}$ is a bounded operator from $H^0(S)$ into $W^\pm(0)$.*
3. *The operators $\frac{1}{2}I + B$ and $\frac{1}{2}I + B'$ are invertible in $H^0(S)$; A is invertible as an operator from $H^0(S)$ into $H^1(S)$; $\frac{1}{2}I + B$ is invertible in $H^1(S)$; $\frac{1}{2}I - B$ and $\frac{1}{2}I - B'$ are Fredholm operators in $H^0(S)$ with zero index.*

Proof. Most of the facts stated in this Proposition can be found in [10] or follow from the theorems on boundedness of singular integral operators on a Lipschitz surface [7]. Namely, the boundedness of the operators $B, B' \colon H^0(S) \to H^0(S)$, $A \colon H^0(S) \to H^1(S)$, $\mathcal{B} \colon H^0(S) \to V^\pm(0)$, and $\mathcal{A} \colon H^0(S) \to W^\pm(0)$ follows from [7]. The invertibility of the operators $\frac{1}{2}I + B$ and $\frac{1}{2}I + B'$ in $H^0(S)$ and the fact that $\frac{1}{2}I - B$ and $\frac{1}{2}I - B'$ are Fredholm in $H^0(S)$ with index zero were proved in the course of study of the problems $N^\pm(0)$ in [10]. (More precisely, the results for $\frac{1}{2}I \pm B$ follow from those obtained in [10] for $\frac{1}{2}I \pm B'$.) In particular, it was proved there that the exterior problem $N^-(0)$ is uniquely solvable in $W^-(0)$ for any $Tu^- \in H^0(S)$.

The Dirichlet problems for $\omega = 0$ were studied in [10] with the use of some other operators of the type of B and B'. The unique solvability of the Dirichlet problems was proved; the solution belongs to $V^\pm(0)$ if $u^\pm \in H^0(S)$ and to $W^\pm(0)$ if $u^\pm \in H^1(S)$.

Using these facts, we shall now easily check the remaining statements of our Proposition. Let us use formula (1.46^-) for solutions from $W^-(0)$ (the discussion of this formula is postponed until the next subsection). It implies that B is a bounded operator in $H^1(S)$. Also, it implies that operator A annihilates only the zero function in $H^0(S)$ (as the same is true for $\frac{1}{2}I + B$). Further, due to formula (1.34^-), which remains valid in Lipschitz domains (as well as ($1.33^\pm$), ($1.34^\pm$), and ($1.37^\pm$)), it is obvious that the formula

$$u = \mathcal{A}(\tfrac{1}{2}I + B')^{-1} Tu^-$$

gives the solution of the exterior problem $N^-(0)$. Thus, we have

$$u^- = A(\tfrac{1}{2}I + B')^{-1} Tu^-, \tag{2.12}$$

which shows that the range of A is the whole space $H^1(S)$. Therefore, A maps $H^0(S)$ continuously and bijectively onto $H^1(S)$, and by Banach's theorem A has a bounded inverse $A^{-1} \colon H^1(S) \to$

$H^0(S)$. By (1.46$^-$), the range of $\frac{1}{2}I + B$ acting in $H^1(S)$ is also $H^1(S)$. Thus, $\frac{1}{2}I + B$ is also invertible in $H^1(S)$. In addition, comparing formulas (1.46$^-$) and (2.12), we obtain (1.99) for $\omega = 0$.

Finally, from two representations of solutions of the interior Dirichlet problem with $u^+ \in H^1(S)$,

$$\mathcal{A}A^{-1}u^+ = \mathcal{B}(\tfrac{1}{2}I + B)^{-1}u^+,$$

it follows that $\mathcal{B}$ acts from $H^1(S)$ into $W^+(0)$. The fact that it also acts from $H^1(S)$ into $W^-(0)$ can be checked by covering an exterior boundary strip by a finite union of simply connected bounded Lipschitz domains adherent to S and using a partition of unity on S.

Theorem 2.5. *For all ω, ω_1, and ω_2:*

1. *The operators $B(\omega)$ and $B'(\omega)$ are bounded in $H^0(S)$, $B(\omega)$ is also bounded in $H^1(S)$, and $A(\omega)$ is bounded as an operator from $H^0(S)$ into $H^1(S)$.*
2. *The operator $\mathcal{B}(\omega)$ is bounded from $H^0(S)$ into $V^\pm(\omega)$ and from $H^1(S)$ into $W^\pm(\omega)$; $\mathcal{A}(\omega)$ is a bounded operator from $H^0(S)$ into $W^\pm(\omega)$.*
3. *The differences $B(\omega_1) - B(\omega_2)$, $B'(\omega_1) - B'(\omega_2)$, and $A(\omega_1) - A(\omega_2)$ are compact operators from $H^0(S)$ into $H^1(S)$; moreover, operators $B(\omega_1) - B(\omega_2)$ are compact in $H^1(S)$.*

Proof. All the statements in parts 1 and 2 referring to $\omega = 0$ are contained in Proposition 2.4. The theorem on the boundedness of singular integral operators from [7] can actually be applied for all ω: it suffices to note that the orders of singularities of the kernels of the differences mentioned in part 3 are at least by two less than those of the singularities of the kernels of $B(\omega_j)$, $B'(\omega_j)$, and $A(\omega_j)$, respectively. This also implies the remaining statements on boundedness and compactness.

We also mention here that, in general, the L_2-integrability on S of the maximal function of a surface potential with a weak singularity follows from the boundedness of the corresponding integral operator of potential type in $L_2(S)$. For example, if $x, z \in S$ and $y \in \Gamma^\pm(x)$ (see (2.3)), then $|y - x| + |y - z| \geqslant |x - z|$; therefore, $C_1|y - z| \geqslant |x - z|$ and

$$\int_S \frac{|f(z)|\,dS_z}{|y - z|} \leqslant C_2 \int_S \frac{|f(z)|\,dS_z}{|x - z|},$$

where the integral in the right-hand side is a bounded integral operator in $L_2(S)$.

2.3. *Integral Formulas*

Theorem 2.6. *Green's formula* (1.39) *is valid for $u \in H^2(G^+)$ and $v \in H^1(G^-)$. Formula* (1.42) *is valid for $u \in W^+(\omega)$ and $v \in H^1(G^+)$, and formula* (1.43) *is valid for $u \in W^+(\omega_1)$ and $v \in W^+(\omega_2)$ without the assumption that the boundary is connected. It follows that the integral representations* (1.44) *and* (1.45) *for solutions from $W^+(\omega)$ and $W^-(\omega)$ are also valid, as well as the relations* (1.46$^\pm$) *on S for these solutions.*

Proof. The first statement is obtained as in [26], Chapter 3, §1, and [15, Section 1.5.3]. To obtain formula (1.42), we use the approximation of G by subdomains G_j with smooth boundaries S_j, which was described in the lemma in [33]. The function v is approximated in $H^1(G^+)$ by functions v_k from $C^1(\overline{G}^+)$ or even from $C^\infty(\overline{G}^+)$, see [15]. Passing to the limit as $j \to \infty$ is allowed due to Lebesgue's dominated convergence theorem and gives formula (1.42) for u and v_k. Now we can take the limit as $k \to \infty$, using the fact that in the case of a Lipschitz domain the trace operator on S is a bounded operator from $H^1(G^+)$ even to $H^{1/2}(S)$ (see [12]). Formula (1.43) is now obtained from (1.42). Cf. [4].

2.4. *The First and the Second Boundary Value Problems. Uniqueness and Regularity*

The problems $D^\pm(\omega)$ and $N^\pm(\omega)$ are posed as in Subsection 1.5, where we take in (1.62$^\pm$) either $f \in H^0(S)$ (and then the solution of $D^\pm(\omega)$ must belong to $V^\pm(\omega)$), or $f \in H^1(S)$ (and then the solution must belong to $W^{\pm}(\omega)$). In ($1.63^{\pm}$), $g \in H^0(S)$, and the solution of $N^{\pm}(\omega)$ must belong to $W^{\pm}(\omega)$. We have mentioned these problems for $\omega = 0$ in Subsection 2.2.

The uniqueness for the exterior problems in $W^-(\omega)$ for all ω and the interior problems in $W^+(\omega)$ for nonreal ω can be verified using Green's formulas.

Theorem 2.7. *Let u be a solution of the problem $D^+(\omega)$ or $D^-(\omega)$ from $V^{\pm}(\omega)$ with $u^{\pm} \in H^1(S)$. Then $u \in W^{\pm}(\omega)$, respectively.*

Proof. For the interior problem, let us consider the volume potential U defined by (2.9). Since its density belongs to $L_2(G^+)$, we have $U \in H^2(\mathbb{R}^3)$ and hence $U^+ \in H^1(S)$. Let us set $v = u + U$. Then v is a solution of system (1.3) with $\omega = 0$ and Dirichlet data from $H^1(S)$. Hence, by [10], $v \in W^+(0)$. Using Corollary 2.3, we conclude that $v \in H^{3/2}(G^+)$, $u \in H^{3/2}(G^+)$, and $u \in W^+(\omega)$.

A slightly more complicated reasoning is required in the exterior case. Let $\theta(x)$ be an infinitely smooth function on $\overline{G}^-$ that is equal to 1 near S and to 0 for $|x| \geqslant R$, where the number R is sufficiently large. Then $(1-\theta)u \in C^\infty(\overline{G}^-)$. Let us take the density F of the potential (2.9) to be $\rho\omega^2 u + L(\partial_x)((1-\theta)u)$. It is easy to see that this function belongs to $L_2(G^-)$ and has a compact support. Hence $U \in H^2(\mathbb{R}^3)$ and $U^- \in H^1(S)$. Let us set $v = \theta u + U$. Then $L(\partial_x)v = 0$ in G^-, $v \in V^-(0)$, $v^- \in H^1(S)$, and thus $v \in W^-(0)$, $v \in H^{3/2}_{\mathrm{loc}}(G^-)$ (by Corollary 2.3), and it follows that $u \in H^{3/2}_{\mathrm{loc}}(G^-)$, $u \in W^-(\omega)$.

Below we call a number ω *exceptional* with respect to the problem D^+ if the homogeneous problem $D^+(\omega)$ has a nontrivial solution in $V^+(\omega)$, or, equivalently, in $W^+(\omega)$. We also call a number ω *exceptional* with respect to the problem N^+ if the homogeneous problem $N^+(\omega)$ has a nontrivial solution in $W^+(\omega)$. We shall relate the exceptional values of ω with the eigenvalues of the variational operators $-L_D$ and $-L_N$ in Subsection 2.7.

2.5. *The Invertibility of the Operators $A(\omega)$, $\frac{1}{2}I \pm B'(\omega)$, and $\frac{1}{2}I \pm B(\omega)$*

Proposition 2.8. *For $\omega'' > 0$, the operators $A(\omega)\colon H^0(S) \to H^1(S)$, $\frac{1}{2}I \pm B(\omega)$, and $\frac{1}{2}I \pm B'(\omega)$ in $H^0(S)$, and $\frac{1}{2}I \pm B(\omega)$ in $H^1(\omega)$ are invertible.*

Proof. Statement 3 of Proposition 2.4 and Statement 3 of Theorem 2.5 show that $A(\omega)\colon H^0(S) \to H^1(S)$, $\frac{1}{2}I \pm B(\omega)$ and $\frac{1}{2}I \pm B'(\omega)$ in $H^0(S)$ are Fredholm operators of index zero for all ω. Let $\omega'' > 0$. Let us prove that $A(\omega)$ and $\frac{1}{2}I \pm B'(\omega)$ annihilate only the function $\varphi = 0$ in $H^0(S)$.

Assume that $A(\omega)\varphi = 0$ and consider $u = \mathcal{A}(\omega)\varphi$. Then $u \in W^{\pm}(\omega)$, $u^{\pm} = 0$, and hence $u = 0$ in $G^{\pm}$ and $\varphi = \partial_\nu u^- - \partial_\nu u^+ = 0$. Here we have used (1.34).

Assume that $(\frac{1}{2}I + B'(\omega))\varphi = 0$ and again consider $u = \mathcal{A}(\omega)\varphi$. Then $Tu^- = 0$, and hence $u = 0$ in G^-, $0 = u^- = u^+$, so that again $u = 0$ in $G^{\pm}$ and $\varphi = 0$. The case $(\frac{1}{2}I - B'(\omega))\varphi = 0$ can be considered in a similar way.

Thus $A(\omega)$ and $\frac{1}{2}I + B'(\omega)$ are invertible. It follows that $\frac{1}{2}I + B^*(\omega)$ and $\frac{1}{2}I \pm B(\omega)$ are also invertible in $H^0(S)$.

To prove that $\frac{1}{2}I \pm B(\omega)$ are invertible in $H^1(S)$, we can use formulas ($1.46^{\pm}$) and Banach's theorem as in the proof of Proposition 2.4. Instead, it is also possible to use formula

$$(\tfrac{1}{2}I \pm B(\omega))^{-1} = A(\omega)(\tfrac{1}{2}I \pm B'(\omega))^{-1}A^{-1}(\omega) \tag{$2.13^{\pm}$}$$

that follows from Proposition 2.10 below.

Corollary 2.9. *For any ω, let*

$$(\tfrac{1}{2}I \pm B(\omega))\varphi = \psi, \tag{2.14}$$

where $\varphi \in H^0(S)$ and $\psi \in H^1(S)$. Then $\varphi \in H^1(S)$. In particular, the operator $\frac{1}{2}I \pm B(\omega)$ has the same null space in $H^0(S)$ and $H^1(S)$.

To prove the corollary, it suffices to act on both sides of (2.14) by, e.g., $(\frac{1}{2}I \pm B(i))^{-1}$ and then apply Statement 3 of Theorem 2.5.

Now let us state the result which contains the analogs of Propositions 1.2, 1.5, and 1.6 for a Lipschitz domain. Here ω is real.

Proposition 2.10. *For all ω, formula* (1.99) *holds on functions from $H^0(S)$, and formula* (1.38) *holds on functions ψ from $H^1(S)$.*

Proof. The proof is similar to that in Subsection 1.7. First, let $\omega' > 0$. For functions $u \in W^+(G)$ we obtain from two representations of Tu^+ in terms of u^+ the relation

$$A^{-1}(\omega)B(\omega)\varphi = B'(\omega)A^{-1}(\omega)\varphi,$$

where $\varphi = u^+$. We set $A^{-1}(\omega)\varphi = \psi$ and multiply both sides by $A(\omega)$ from the left. (Instead of the relation between u^+ and Tu^+ we could use the relation between u^- and Tu^-.) It remains to pass to the limit as $\omega' \to 0$.

Formula (1.38) is obtained similarly: see the proof of Proposition 1.9.

Theorem 2.11. *In the space $H^0(S)$,*

$$\operatorname{Ker} A(\omega) = \{Tu^+ : u \in W^+(\omega),\, u^+ = 0\} = \operatorname{Ker}(\tfrac{1}{2}I + B'(\omega)). \tag{2.15}$$

Furthermore, $\operatorname{Ker}(\frac{1}{2}I - B(\omega))$ *in $H^0(S)$ is contained in $H^1(S)$ and is expressed by the formula*

$$\operatorname{Ker}(\tfrac{1}{2}I - B(\omega)) = \{u : u \in W^+(\omega),\, Tu^+ = 0\}. \tag{2.16}$$

The proofs essentially repeat the proofs of Propositions 1.2, 1.5, and 1.6. For example, let us check (2.16). If $u \in W^+(\omega)$ and $Tu^+ = 0$, then from (1.46^+) it follows that $(\frac{1}{2}I - B(\omega))u^+ = 0$. Conversely, let $(\frac{1}{2}I - B(\omega))\varphi = 0$, $\varphi \in H^0(S)$. Then $\varphi \in H^1(S)$ by Corollary 2.9. Set $u = \mathcal{B}(\omega)\varphi$. Then $u \in W^\pm(\omega)$ and $u^- = 0$ by (1.37^-), $u = 0$ in G^-, and $0 = Tu^- = Tu^+$ by (1.38). Using ($1.37^\pm$), we see that $u^+ = \varphi$. $\square$

As ω is here actually real, it is clear that all the null spaces coincide with their complex conjugate subspaces.

Theorem 2.12.

1. *For any ω the following statements are equivalent*:
   (a) *ω is not exceptional with respect to the problem D^+*;
   (b) *the operator $A(\omega)\colon H^0(S) \to H^1(S)$ is invertible*;
   (c) *the operator $\frac{1}{2}I + B'(\omega)\colon H^0(S) \to H^0(S)$ is invertible*;
   (d) *the operator $\frac{1}{2}I + B(\omega)\colon H^0(S) \to H^0(S)$ is invertible*;
   (e) *the operator $\frac{1}{2}I + B(\omega)\colon H^1(S) \to H^1(S)$ is invertible.*
2. *For any ω the following statements are also equivalent*:
   (f) *ω is not exceptional with respect to the problem N^+*;
   (g) *the operator $\frac{1}{2}I - B'(\omega)\colon H^0(S) \to H^0(S)$ is invertible*;
   (h) *the operator $\frac{1}{2}I - B(\omega)\colon H^0(S) \to H^0(S)$ is invertible*;
   (i) *the operator $\frac{1}{2}I - B(\omega)\colon H^1(S) \to H^1(S)$ is invertible.*

Proof. We use Proposition 2.4 and Theorem 2.5. Since $A(\omega)$ is a Fredholm operator from $H^0(S)$ into $H^1(S)$ with zero index for all ω, the statement (a), i.e., the triviality of (2.15), is equivalent to (b). Similarly, with the account of (2.15), (a) is equivalent to (c). The operators $\frac{1}{2}I + B(\omega)$, $\frac{1}{2}I + B^*(\omega)$ and (since $B^*(\omega) = \overline{B}'(\omega)$) $\frac{1}{2}I + B'(\omega)$ are invertible in $H^0(S)$ for the same values of ω. In particular, (c) is equivalent to (d).

Using Proposition 2.10, we see that (e) follows from (a) and (c): see (2.13^+). Conversely, from (e) it follows that $\mathrm{Ker}(\frac{1}{2}I + B(\omega)) = \{0\}$ in $H^0(S)$ (see Corollary 2.9) and hence $\frac{1}{2}I + B(\omega)$ is invertible in $H^0(S)$. We see that (a)–(e) are equivalent.

Now (g) and (h) are obviously equivalent, and (h) is equivalent to (f) (since $\frac{1}{2}I - B(\omega)$ is Fredholm in $H^0(S)$ with index zero and its null space lies in $H^1(S)$; see (2.16)). From (i) we obtain (h) (as (d) from (e)). It remains to derive (i) from (h) (avoiding the assumption that $A^{-1}(\omega)$ exists). For this we use the following: $\frac{1}{2}I - B(\omega)$ is a bounded operator in $H^1(S)$, from (h) it follows that this operator is one-to-one, and from Corollary 2.9 and (h) it follows that its image is $H^1(S)$. Thus $\frac{1}{2}I - B(\omega)$ is invertible in $H^1(S)$ again by Banach's theorem, and we see that (f)–(i) are equivalent.

2.6. *Formulas for Solutions of Main Boundary Value Problems*

Theorem 2.13. *If ω is not exceptional with respect to the problem D^+, then formulas* (1.93) *and* (1.96) *hold with $u^+ \in H^0(S)$ or $u^+ \in H^1(S)$ and $Tu^- \in H^0(S)$, respectively, and formulas* (1.94) *and* (1.95) *hold with $u^\pm \in H^1(S)$. If ω is not exceptional with respect to the problem N^+, then formula* (1.97) *holds with $Tu^+ \in H^0(S)$ and formula* (1.98) *holds with $u^- \in H^0(S)$ or $u^- \in H^1(S)$.*

For $\omega'' > 0$, some of these facts where already used in the proof of Proposition 2.10.

For exceptional values of ω we can use the formulas mentioned after (1.98). We leave out the details.

Proposition 2.14. *Let ω be not exceptional with respect to the problem D^+. Then equalities* (1.104) *hold. If, in addition, ω is not exceptional with respect to the problem N^+, then equalities* (1.107) *hold as well.*

2.7. *Exceptional Frequencies ω and Eigenvalues of L_D and L_N*

The operators L_D and L_N should now be understood in the variational sense. To define $-L_D$, consider the sesquilinear form (cf. (1.39))

$$a_D[u,v] = \int_{G^+} E(u,\overline{v})\,dx \quad \text{on } H_0^1(G^+), \tag{2.17}$$

where, as before, $H_0^1(G^+)$ is the closure of $C_0^\infty(G^+)$ in $H^1(G^+)$. The corresponding quadratic form is positive definite due to the Gårding inequality

$$\varepsilon\|u\|^2_{1,G^+} \leqslant \int_{G^+} E(u,\bar{u})\,dx \qquad (u \in H_0^1(G^+)) \tag{2.18}$$

(cf. (1.71)), which is still true in any Lipschitz (and even in any bounded) domain (cf. [27, Chapter 1]). Thus $H_0^1(G^+)$ can be considered as a Hilbert space with the inner product (2.17). This form is closed and densely defined in $L_2(G^+)$, and it defines a selfadjoint operator A_D with domain $D(A_D) \subset H_0^1(G^+)$ (see, e.g., [19, Chapter 6]). This operator has a bounded inverse B_D defined by

$$\int_{G^+} w \cdot \overline{v}\,dx = a_D[B_D w, v] \qquad (w \in H^0(G^+),\ v \in H_0^1(G^+)), \tag{2.19}$$

and the domain $D(A_D)$ of A_D coincides with the range of B_D. We now define $-L_D$ as A_D.

The operator $A_N = I - L_N$ is defined in the same manner by the sesquilinear form

$$a_N[u,v] = \int_{G^+} [E(u,\overline{v}) + u\cdot\overline{v}]\,dx \quad \text{on } H^1(G^+). \tag{2.20}$$

The corresponding quadratic form is positive definite due to the analog of the Gårding inequality

$$\varepsilon\|u\|^2_{1,G^+} \leqslant \int_{G^+} E(u,\bar{u})\,dx + \|u\|^2_{0,G^+} \qquad (u \in H^1(G^+)) \tag{2.21}$$

(cf. (1.76)), which works on Lipschitz domains (cf. [27, Chapter 1]), and the form (2.20) can be considered as an inner product on $H^1(G^+)$. This form is closed and densely defined in $H^0(G^+)$, and it defines a selfadjoint operator A_N in $H^0(G^+)$ with domain $D(A_N) \subset H^1(G^+)$. It has a bounded inverse B_N such that

$$\int_{G^+} w \cdot \overline{v}\, dx = a_N[B_N w, v] \qquad (w \in H^0(G^+),\ v \in H^1(G^+)). \tag{2.22}$$

The spectra of selfadjoint operators A_D and A_N are real and discrete, since the operators B_D and B_N are compact in view of the compactness of the embedding of $H^1(S)$ into $H^0(S)$.

Theorem 2.15. *The set of values of $\rho\omega^2$ for which ω is exceptional with respect to the problem D^+ coincides with the spectrum of $-L_D$, and the null spaces* $\operatorname{Ker} A(\pm\omega)$ *coincide with the corresponding eigenspace of $-L_D$.*

Similarly, the set of values of $\rho\omega^2$ for which ω is exceptional with respect to the problem N^+ coincides with the spectrum of $-L_N$, and the null spaces $\operatorname{Ker}[\frac{1}{2}I - B(\pm\omega)]$ *coincide with the corresponding eigenspace of $-L_N$.*

The eigenfunctions of both operators belong to $H^{3/2}(G^+)$ in any bounded Lipschitz domain G^+.

Proof. Let u be any nontrivial solution of system (1.3) from $W^+(\omega)$ with $u^+ = 0$. Let us take $v \in H_0^1(G^+)$ and apply Green's first formula (1.42) to u and $\overline{v}$. We obtain

$$\rho\omega^2 \int_{G^+} u \cdot \overline{v}\, dx = \int_{G^+} E(u, \overline{v})\, dx. \tag{2.23}$$

We see that $B_D u = (\rho\omega^2)^{-1} u$, i.e. $-L_D u = \rho\omega^2 u$.

Conversely, let u be an eigenfunction of $-L_D$ corresponding to an eigenvalue $\rho\omega^2$. Then $u \in H_0^1(G^+)$, and we have (2.23). This implies that $L_D u + \rho\omega^2 u = 0$ in G^+ in the sense of distributions and, therefore, in the usual sense. In particular, $u(x)$ is a solution of the homogeneous problem $D^+(\omega)$ in $H^{1/2}(G^+)$, and thus $u \in V^+(\omega)$ (by Theorem 2.1), $u \in W^+(\omega)$ (by Theorem 2.7), and, finally, $u \in H^{3/2}(G^+)$ (by Corollary 2.3).

Now let $u(x)$ be any nontrivial solution of system (1.3) from $W^+(\omega)$ with $Tu^+ = 0$. Let us take $v \in H^1(G^+)$ and apply Green's first formula (1.42) to u and $\overline{v}$. We again obtain (2.23) and conclude that u is an eigenfunction of $-L_N$ corresponding to the eigenvalue $\rho\omega^2$.

Conversely, let u be an eigenfunction of $-L_N$ corresponding to an eigenvalue $\rho\omega^2$. Then $u \in H^1(G^+)$ and (2.23) is valid for all $v \in H^1(G^+)$. Using a function v from $C_0^\infty(G^+)$, we see that u is a solution of (1.3) in G^+ in the sense of distributions and, therefore, in the usual sense. Let us introduce the volume potential

$$U_1(x) = \int_{G^+} \mathcal{E}(x - y, i)\rho(\omega^2 + 1)u(y)\, dy. \tag{2.24}$$

This function belongs to $H^2(G^+)$ and satisfies the equation $(L - \rho)U_1 = \rho(\omega^2 + 1)u$ in G^+, and $TU_1^+ \in H^{1/2}(S)$. Set $u_1 = u + U_1$. We have $(L - \rho)u_1 = 0$ in G^+ and $u_1 \in H^1(G^+)$, and also $(u_1)_*^+ \in L^2(S)$ by Theorem 2.1. Furthermore, we have

$$\int_{G^+} (L - \rho)U_1 \cdot \overline{v}\, dx = \rho(\omega^2 + 1)\int_{G^+} u \cdot \overline{v}\, dx = \int_{G^+} E(u, \overline{v})\, dx + \rho \int_{G^+} u \cdot \overline{v}\, dx$$

by (2.23), and by Green's formula (1.39),

$$\int_{G^+} (L - \rho)U_1 \cdot \overline{v}\, dx = -\rho \int_{G^+} U_1 \cdot \overline{v}\, dx - \int_{G^+} E(U_1, \overline{v})\, dx + \int_S TU_1^+ \cdot \overline{v} dS.$$

Hence

$$\int_{G^+} E(u_1,\overline{v}) + \rho \int_{G^+} u_1\overline{v}\,dx = \int_S TU_1^+ \cdot \overline{v}\,dx$$

for any $v \in H^1(G^+)$. Let us now define u_2 as the only solution of $(L-\rho)u_2 = 0$ in $W^+(i)$ (i.e., in $H^{3/2}(G^+)$) with the boundary condition $Tu_2^+ = TU_1^+$. With the same choice of v, we get

$$0 = \int_{G^+} (L-\rho)u_2 \cdot \overline{v}\,dx = -\rho \int_{G^+} u_2 \cdot \overline{v}\,dx - \int_{G^+} E(u_2,\overline{v})\,dx + \int_S TU_1^+ \cdot \overline{v}\,dx.$$

Comparing the last relations, we see that

$$\int_{G_+} E(u_1-u_2,\overline{v})\,dx + \rho \int_{G^+} (u_1-u_2)\cdot \overline{v}\,dx = 0.$$

Setting $v = u_1 - u_2$, we conclude that $u_1 = u_2$. Now we see that $u = u_2 - U_1$ belongs to $H^{3/2}(G^+)$ and $W^+(\omega)$ (by Corollary 2.3), and that $Tu^+ = 0$ in the usual sense for this space.

For both variational problems the spectral asymptotics are the same as in the smooth case (see [5]).

2.8. *Reduction of Problems* I–IV *to Integral Equations on S for Nonexceptional* ω

Here we can repeat word for word all the constructions of Subsection 1.4 with the following specification: the solutions of the boundary value problems are considered in $W^{\pm}(\omega)$. All the operators $A(\omega)$, $T^{\pm}(\omega)$, $T(\omega)$ are bounded from $H^0(S)$ into $H^1(S)$.

2.9. *Spectral Properties of Operator* $A(\omega)$

In this and the following subsections we shall show that somewhat coarser forms of the results of Subsections 1.8–1.10 remain valid in the case of a Lipschitz surface S.

Theorem 2.16. *Statements* 1 *and* 3 *of Theorem* 1.11 *remain true.*

These statements are obtained exactly as in the case of a smooth S.

Instead of Statement 2 of Theorem 1.11, we can use only a weaker statement, namely a part of the Statement 3 of Theorem 2.5: $A(\omega) - A(i)$ is a compact operator from $H^0(S)$ into $H^1(S)$, so that $A^{-1}(i)(A(\omega)-A(i))$ is a compact operator in $H^0(S)$. However, this implies that on a Lipschitz surface S the operator $A(\omega)$ may be considered as a weak perturbation of a selfadjoint negative operator $A(i)$ or $A(\omega_1)$ with any pure imaginary ω_1.

As in the case of a smooth S, for $\omega' \neq 0$ the only possible real eigenvalue of $A(\omega)$ is zero, the corresponding root space is finite-dimensional and contains only eigenfunctions.

Theorem 1.12 is replaced by the following:

Theorem 2.17. *For any ω the characteristic numbers $z_j(A(\omega))$ tend to $-\infty$ and satisfy the inequality*

$$|z_j(A(\omega))| \geqslant Cj^{1/2}, \tag{2.25}$$

where C is a positive constant. If the surface S is almost smooth, then

$$z_j(A(\omega)) = -c(A)j^{1/2} + o(j^{1/2}) \qquad (j\to\infty), \tag{2.26}$$

where the constants $c_j(A)$ are independent of ω.

If $\omega' \neq 0$, then $\operatorname{Im} z_j/\operatorname{Re} z_j \to 0$ as $j\to\infty$. The signs of $\operatorname{Im} z_j$ coincide with the sign of ω'.

For $|\omega'| < \omega''$, $\operatorname{Re} z_j$ are negative.

Estimate (2.25) follows from the fact that $A(\omega)$ is bounded as an operator from $H^0(S)$ into $H^1(S)$ (see references in [2] or estimates of s-numbers in [3] and [14, Chapter II]). Asymptotics (2.26) for an almost smooth S is obtained in [3]. The other statements follow from the fact that $A(\omega)$ is a weak perturbation of $A(i)$ (cf. [14], [1], and [2]), and from (1.113) and (1.118).

Theorem 2.18. *For* $\operatorname{Im}\omega \neq 0$ *the root functions of* $A(\omega)$ *are complete in* $H^0(S)$*. The root functions corresponding to nonzero eigenvalues belong to* $H^1(S)$*,*[2] *and if* 0 *is not an eigenvalue, then we also have the completeness in* $H^1(S)$*. The Fourier series of* $f \in H^0(S)$ *with respect to a complete minimal system of the root functions admits the summability to* f *in* $H^0(S)$ *by the Abel–Lidskiĭ method of order* $2+\varepsilon$ *with arbitrarily small* ε*, and if* 0 *is not an eigenvalue, then the same is true in* $H^1(S)$*.*

Here we once more use the fact that $A(\omega)$ is a weak perturbation of $A(i)$ (cf. [14]). The description of the Abel–Lidskiĭ summability method and the corresponding theorem which we use can be found in [23], [1], [2], and [4]. The theorem applies to a weak perturbation of a selfadjoint operator and describes a method of reconstructing of any function from its Fourier series with respect to the root functions of this operator.

2.10. *Spectral Properties of Operators* $T^{\pm}(\omega)$ *and* $T(\omega)$

As in Subsections 1.9–1.10, here we assume that ω is not exceptional with respect to the problem N^+ when considering the operator $T^+(\omega)$, and with respect to the problem D^+ when considering the operator $T^-(\omega)$.

Theorem 2.19. *Statements* 1 *and* 2 *of Theorem* 1.15 *remain true.*

Instead of Statement 3 of Theorem 1.15 we have the compactness of operators $T^{\pm}(\omega) - T^{\pm}(i) : H^0(S) \to H^1(S)$, from which it follows that the operators $(T^{\pm}(i))^{-1}(T^{\pm}(\omega) - T^{\pm}(i))$ are compact in $H^0(S)$.

As in the case of a smooth S, $T^+(\omega)$ can have the eigenvalue zero simultaneously with $A(\omega)$, and then $\operatorname{Ker} T^+(\omega) = \operatorname{Ker} A(\omega)$ (and ω is real). The operator $T^-(\omega)$ cannot have the eigenvalue zero.

Theorem 2.20. *For any* ω *the real parts of the characteristic numbers* $z_j(T^{\pm}(\omega))$ *of the operators* $T^{\pm}(\omega)$ *tend to* $-\infty$*, and*

$$|z_j(T^{\pm}(\omega))| \geqslant Cj^{1/2} \tag{2.27}$$

with positive constant C*. If the surface* S *is almost smooth, then*

$$z_j(T^{\pm}(\omega)) = -c(T^{\pm})j^{1/2} + o(j^{1/2}) \qquad (j \to \infty), \tag{2.28}$$

where positive coefficients $c(T^{\pm})$ *are independent of* ω*.*

If the operators are nonselfadjoint, then $\operatorname{Im} z_j / \operatorname{Re} z_j \to 0$ *as* $j \to \infty$*. The signs of* $\operatorname{Im} z_j$ *coincide with the sign of* ω'*.*

For $|\omega'| < \omega''$*, the real parts of* z_j *are negative.*

Theorem 2.21. *If the operator* $T^+(\omega)$ *or* $T^-(\omega)$ *is nonselfadjoint, then its root functions belong to* $H^1(S)$ *and are complete in* $H^0(S)$ *and* $H^1(S)$*. The Fourier series of a function* $f \in H^j(S)$ *with respect to the root functions admits the summability by the Abel–Lidskiĭ method of order* $2+\varepsilon$ *with arbitrarily small* $\varepsilon > 0$ *to* f *in* $H^j(S)$ $(j = 0, 1)$*.*

If $T^+(\omega)$ is selfadjoint, its eigenfunctions corresponding to nonzero eigenvalues belong to $H^1(S)$. If $T^-(\omega)$ is selfadjoint, all its eigenfunctions belong to $H^1(S)$.

The proofs are similar to those given in Subsection 2.9. The only difference is that in order to obtain formula (2.28) in the case of an almost smooth S we use two representations for $T^{\pm}(\omega)$ which follow from (1.99):

$$T^{\pm}(\omega) = (\tfrac{1}{2}I \mp B(\omega))^{-1}A(\omega) = A(\omega)(\tfrac{1}{2}I \mp B'(\omega))^{-1};$$

this is essential for the application of the result of [3].

Finally, we can describe the spectral properties of $T(\omega)$ under the assumption that ω is not exceptional for the problems D^+ and N^+. However, these properties are similar to those of $T^-(\omega)$; therefore we do not repeat the formulations.

[2] Cf. [4].

2.11. *Problems* I–IV *for Exceptional* ω

The ways of reduction of these problems to equations on nonsmooth S are similar to those described in the case of a smooth S in Subsection 1.11, and the properties of the corresponding operators are similar to those described in two previous subsections.

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